THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The two hundred sixteenth regular meeting of the Society was held at Columbia University on Saturday, April 23, 1921, extending through the usual morning and afternoon sessions. The attendance included the following sixty-seven members:

Professor R. C. Archibald, Professor R. A. Arms, Dr. Charlotte C. Barnum, Professor A. A. Bennett, Professor E. G. Bill, Professor G. D. Birkhoff, Professor H. F. Blichfeldt, Dr. R. F. Borden, Professor R. W. Burgess, Professor Abraham Cohen, Dr. G. M. Conwell, Professor Louise D. Cummings, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Jesse Douglas, Professor L. P. Eisenhart, Professor W. B. Fite, Mr. R. M. Foster, Mr. Philip Franklin, Mr. B. P. Gill, Dr. T. H. Gronwall, Dr. C. C. Grove, Dr. C. M. Hebbert, Professor E. R. Hedrick, Lieutenant R. S. Hoar, Professor L. S. Hulburt, Professor W. A. Hurwitz, Mr. S. A. Joffe, Professor Edward Kasner, Professor O. D. Kellogg, Dr. E. A. T. Kircher, Professor J. R. Kline, Mr. Harry Langman, Professor Florence P. Lewis, Mr. L. L. Locke, Mr. John McDonnell, Professor H. F. MacNeish, Professor L. C. Mathewson, Professor H. H. Mitchell, Professor G. W. Mullins, Dr. Almar Naess, Professor E. J. Oglesby, Professor W. F. Osgood, Professor F. W. Owens, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor L. W. Reid, Dr. C. N. Reynolds, Mr. L. H. Rice, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. J. E. Rowe, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor L. L. Silverman, Professor P. F. Smith, Dr. J. M. Stetson, Miss Louise E. C. Stuerm, Professor K. D. Swartzel, Mr. H. S. Vandiver, Professor J. H. M. Wedderburn, Professor Mary E. Wells, Dr. Norbert Wiener, Miss Ella C. Williams, Professor H. S. White, Professor J. K. Whittemore, Professor Ruth G. Wood.

Professors L. E. Dickson and W. F. Osgood presided at the morning session, and Professor P. F. Smith, relieved by Professors H. S. White and Abraham Cohen, in the afternoon. The afternoon session opened with a one-hour lecture given by Professor W. A. Hurwitz at the invitation of the programme committee, on *Topics in the theory of divergent series.*

The Council announced the election of the following persons to membership in the Society:

Mr. Lewis Albert Anderson, Central Life Assurance Society, Des Moines;
Dr. Eugene Manasseh Berry, Purdue University;
Mr. Raymond Van Arsdale Carpenter, Metropolitan Life Insurance Company, New York;
Mr. James Douglas Craig, Metropolitan Life Insurance Company, New York;
Mr. Bernard Francis Dostal, Oberlin College;
Mr. George Graham, Central States Life Insurance Company, St. Louis;
Dr. Jacob Millison Kinney, Hyde Park High School, Chicago;
Mr. John Ruse Larus, Jr., Phoenix Mutual Life Insurance Company, Hartford;
Professor Daniel Acker Lehman, Goshen College;
Mr. Joseph Brotherton Maclean, Mutual Life Insurance Company, New York;
Mr. Franklin Bush Mead, Lincoln National Life Insurance Company, Fort Wayne;
Professor James Newton Michie, University of Texas;
Mr. Henry Moir, Home Life Insurance Company, New York;
Mr. William Oscar Morris, North American Life Insurance Company, Chicago;
Mr. Oliver Winfred Perrin, Penn Mutual Life Insurance Company, Philadelphia;
Professor Harris Rice, Worcester Polytechnic Institute;
Miss Jessie M. Short, Reed College;
President Wen Shion Tsu, Nanyang Railway and Mining College;
Professor Buz M. Walker, Mississippi Agricultural and Mechanical College;
Mr. Robert Montague Webb, Kansas City Life Insurance Company;
Mr. Archibald Ashley Welch, Phoenix Mutual Life Insurance Company, Hartford;
Professor Frank Edwin Wood, Michigan Agricultural College;
Mr. Joseph Hooker Woodward, Equitable Life Assurance Society, New York;
Mr. William Young, New York Life Insurance Company.

Eleven applications for membership were received.

It was voted by the Council to accept the invitations received at the preceding meeting to hold the next annual meeting of the Society at Toronto during the Christmas holidays in connection with the meeting of the American Association for the Advancement of Science, and to hold a meeting of the Society with the Chicago Section at Chicago in 1922. The time of the Chicago meeting was not determined.

Professor G. D. Birkhoff was reelected a member of the Editorial Committee of the TRANSACTIONS for a term of three years beginning October 1, 1921.

Professor P. F. Smith presented to the Council the recommendations of the committee authorized at the last meeting to consider policies for conserving the interests of the Society in foreign countries for the year 1921. These recommendations

were adopted and the committee discharged with the thanks of the Council. It was voted to request the President to appoint a committee to consider and report to the Council a policy for 1922 with regard to these and related matters. A committee was also authorized to prepare nominations for officers and other members of the Council to be elected at the annual meeting in December.

It was decided that on the request of the contributor twenty-five reprints (with covers) of articles in the BULLETIN be given without charge, and that a statement be made to authors concerning the cost of reprints with a request that they forego this privilege of free copies unless they can make good use of them.

The usual luncheon and dinner were held at the Faculty Club.

The titles and abstracts of the papers read at this meeting follow below. Professor Huntington's paper was read by Professor Kellogg, and Professor Schwatt's papers by Professor Kline; the papers of Professors Poor, Lipka, Jackson, and McMahon, Dr. Walsh, Professor Fischer, Dr. Post and Professor Coble were read by title.

1. Professor W. F. Osgood: *On the gyroscope.*

When a gyroscope moves under the action of any forces, its axis describes a cone at a definite rate. The discovery of a form of the intrinsic relation between these three factors is the chief result of this paper. Since one point, O, in the axis is fixed, the applied forces (the constraint at O being omitted) can be replaced (a) by a single force, $\mathfrak{F}$, acting at a point ζ of the axis at unit distance from O and at right angles to $O\zeta$; (b) by a couple, $\mathfrak{N}$, whose plane is perpendicular to the axis. The contribution of the cone, $\mathfrak{K}$, lies in its *bending*, κ. By this is meant the rate at which the tangent plane at O turns when ζ describes its path, $\mathfrak{C}$, on the unit sphere with unit velocity. Finally, let v denote the velocity with which ζ describes its path $\mathfrak{C}$ in the case of the actual motion, and let r denote the angular velocity of the gyroscope about its axis. Then

$$Av\frac{dv}{ds} = T, \qquad A\kappa v^2 + Crv = Q, \qquad C\frac{dr}{dt} = N,$$

where s denotes the arc of $\mathfrak{C}$, T and Q the components of $\mathfrak{F}$ along the tangent to $\mathfrak{C}$ and the normal perpendicular to $O\zeta$

respectively, and C and A are the moments of inertia about the axis $O\zeta$ of the gyroscope and an axis through O perpendicular to $O\zeta$. The foregoing relations are *intrinsic*, i.e. independent of any particular choice of coordinates for the gyroscope. They apply directly to the motion of the axis. In the case most important in practice, $N = 0$, and the angular velocity, r, reduces to a constant, ν. There remain, then, merely the first two equations, in which r is replaced by ν.

2. Professor H. S. White: *Seven points in space and the eighth associated point.*

The construction given by O. Hesse in CRELLE'S JOURNAL, vol. 20, is interwoven with the relations of the eight points to three quadric surfaces on which they lie, and to one auxiliary quadric. Using the same construction by points and lines, this paper avoids all reference to quadrics. By algebraic formula in invariant form it proves the uniqueness of the derived eighth point and the symmetry of the completed set of eight, and it gives explicitly the equation of the derived point, covariant, of degree seven, in the given points.

3. Professor L. E. Dickson: *Most general composition of polynomials.*

This paper treats the problem to find three polynomials $f(x) \equiv f(x_1, \cdots, x_n)$, ϕ, F such that $f(x)\phi(\xi) = F(X)$, identically in $x_1, \cdots, x_n, \xi_1, \cdots, \xi_n$, when $X_1, \cdots, X_n$ are bilinear functions of these $2n$ independent variables. The author proves that this problem reduces to the case of a single polynomial f for which $f(x)f(\xi) = f(X)$. This case was treated by him in the COMPTES RENDUS DU CONGRÈS INTERNATIONAL DES MATHÉMATICIENS, Strasbourg, 1920, pp. 131–146. The new paper will appear in the COMPTES RENDUS.

4. Professor L. E. Dickson: *Number of real roots by Descartes's rule of signs.*

In this paper, the author gives a complete, elementary proof of Descartes's rule of signs for the number of real roots of an equation $f(x) = 0$ with real coefficients. The proof follows from the following fact, which is proved by induction. The number of variations of signs in the set $a_0, a_1, \cdots, a_n$ exceeds by a positive even integer the number of variations of signs in $b_0, \cdots, b_{n-1}$, if b_0 is any chosen number of the same

sign as a_0, while the remaining b's are found as in the process of synthetic division by means of a positive multiplier.

5. Professor L. P. Eisenhart: *The Einstein solar field.*

This paper appears in full in the present number of this BULLETIN.

6. Professor J. K. Whittemore: *A special kind of ruled surface.*

In this paper are determined all ruled surfaces having the property that every pair of curved asymptotic lines cuts a constant length from a variable ruling. It is shown that these surfaces have an analogy with the right helicoid; in particular that all rulings of such a surface are parallel to a plane and that the parameter of distribution of such a surface is constant. It is proved that a surface of this type is completely determined by the choice of a plane curve, a straight line, and the constant parameter of distribution. Certain questions regarding these surfaces are considered.

7. Professor V. C. Poor: *On the theorems of Green and Gauss.*

This paper contains a definition for the divergence of a vector in terms of the point differential, namely:

$$\operatorname{div} u\, dP \times \delta P \wedge \vartheta P = du \times \delta P \wedge \vartheta P + \delta u \times \vartheta P \wedge dP + \vartheta u \times dP \wedge \delta P,$$

with the proof for the uniqueness and the existence of the idea so defined. The statement and proof of the following very general form of Green's theorem is given:

$$\int I_1 \left(K\alpha \frac{d \operatorname{grad} \beta}{dP} - K\beta \frac{d \operatorname{grad} \alpha}{dP} \right) d\tau = \int (K\alpha \operatorname{grad} \beta - K\beta \operatorname{grad} \alpha) \times u d\sigma.$$

Another theorem states that

$$\int \left(\frac{d\alpha}{dP} u \right) x d\tau = - \int u \times u \cdot \alpha x d\sigma - \int \alpha \frac{dx}{dP} u d\tau.$$

8. Dr. J. E. Rowe: *Pressure distribution around a breech-block.*

The breech-block of a certain gun is circular in form and is held in place by a thread on its circumference. When the

gun is fired the gas presses against a circular portion of this breech-block, whose area is equal to the cross-sectional area of the powder chamber of the gun, and which is eccentrically situated with respect to the breech-block. The purpose of the paper is to derive the intensity of the pressure on any part of the thread from a given powder pressure.

9. Professor E. V. Huntington: *The mathematical theory of proportional representation. Third paper.*

The author's method of apportionment of representatives, which was presented at the December and February meetings, and which is now known as the "method of equal proportions," solves the problem of apportionment by a direct and obvious comparison between the several states, without the use of the idea of total error. A satisfactory expression for total or average error may, however, be stated, as follows: $E = \sqrt{S/N}$, where $S = \Sigma[(a - \alpha)^2/a]$. Here a is the actual and α the theoretical number of representatives assigned to a typical state A, and N is the total number of representatives. This quantity E (or S) is the quantity which is minimized by the method of equal proportions. The quantity Q, proposed at the February meeting by Professor F. W. Owens, and minimized by the Willcox method of major fractions, may be written as $Q = \Sigma[(a - \alpha)^2/\alpha]$. These quantities S and Q and other similar expressions are compared, and reasons given for preferring S.

10. Professor F. W. Owens: *On the apportionment of representatives. Second paper.*

This paper is supplementary to the author's paper on the same subject read at the February meeting of the Society and contains further developments and illustrations of the results of the different methods.

11. Professor Joseph Lipka: *On the geometry of motion in a curved space of n dimensions.*

In this paper are derived a number of geometric properties of certain systems of ∞^{2n-1} curves, termed q systems, in a curved space of n dimensions. These systems include dynamical trajectories, brachistochrones, catenaries, and velocity curves, under conservative forces or under arbitrary positional fields of force as special cases. The properties are

expressed in terms of osculating and hyperosculating geodesic surfaces and hyperosculating geodesic circles and the loci of their centers of curvature.

12. Professor Dunham Jackson: *Note on an irregular expansion problem.*

This paper will appear in full in the January number of this BULLETIN.

13. Professor James McMahon: *Hyperspherical goniometry, with applications to the theory of correlation for n variables.*

The chief suggestion for this paper is contained in an article by Karl Pearson in BIOMETRIKA (vol. 11, p. 237). In Part I, the formulas of spherical trigonometry are generalized for the hypersphere in n dimensions, the underlying hypergeometry being first sketched in a form adapted to the purpose in hand. In Part II, a far-reaching connection between the geometry of the hypersphere and the theory of correlation for n variables is established by linearly transforming the 'ellipsoids' of equal frequency (on an n-dimensional correlation chart) into hyperspherical surfaces. The generalized formulas of hyperspherical goniometry are then applied to furnish easy proofs of various new and old theorems in multiple and partial correlation (including the relations referred to above), and to point the way to further developments.

14. Dr. J. L. Walsh: *On the location of the roots of polynomials.*

This paper contains a proof of the following theorem: If the coefficients of a polynomial $f(z)$ are linear in each of the parameters $\alpha_1, \alpha_2, \cdots, \alpha_k$ and are symmetric (but not necessarily homogeneous) in these parameters, and if the points α lie in a circular region C, then for any value of z we may replace the α's by k parameters *which coincide in* C and this without changing the value of $f(z)$.

This theorem has various applications; it can be easily proved that if all the k roots of a polynomial $\varphi(z)$ lie on or within a circle C whose center is α and radius r, then all the roots of

$$A_0\varphi(z) + A_1\varphi'(z) + A_2\varphi''(z) + \cdots + A_k\varphi^{(k)} = 0$$

(when the A's are arbitrary constants) lie on or within the circles of radius r whose centers are the roots of

$$A_0(z-\alpha)^k + kA_1(z-\alpha)^{k-1} + k(k-1)A_2(z-\alpha)^{k-2} + \cdots + k(k-1)\cdots 1\cdot A_k = 0.$$

15. Professor C. A. Fischer: *The kernel of the Stieltjes integral corresponding to a completely continuous transformation.*

In the 1918 ACTA MATHEMATICA, F. Riesz has proved that a linear transformation which is completely continuous, $A[f]$, can be decomposed into the sum of two orthogonal transformations, $A_1[f]$ and $A_2[f]$, such that the transformation $B_1[f] = f(x) - A_1[f]$ has an inverse, and the equations $B_2^n[\varphi] = 0$ and $B^n[\varphi] = 0$ have the same solutions. He has also proved in an earlier paper that every linear transformation can be put into the form

$$A[f] = \int_a^b f(y)d_yK(x, y).$$

In this BULLETIN, October, 1920, Professor Fischer gave the conditions which $K(x, y)$ must satisfy in order that $A[f]$ be completely continuous. In the present paper he has proved that the function $K_2(x, y)$, corresponding to Riesz' $A_2[f]$, can be put into the form

$$K_2(x, y) = \sum_{i=1}^{n} \varphi_i(x)\psi_i(y),$$

where the functions $\varphi_i(x)$ are solutions of the equation $B^\nu[\varphi] = 0$, ν being a positive integer such that every solution of $B^{\nu+1}[\varphi] = 0$ is also a solution of $B^\nu[\varphi] = 0$. The applications of this theorem to Stieltjes' integral equations of both the first and second kinds are then discussed.

16. Dr. E. L. Post: *On a simple class of deductive systems.*

In the present paper the author considers a general class of deductive systems involving primitive functions of but one argument, and solves for all such systems the following problem: to find a method for determining in a finite number of steps whether a given enunciation of the system can or cannot be asserted by means of the postulates of the system. The solution is obtained by directly analyzing the way in which assertions are generated from the primitive assertions by the rules of deduction of the system, and gives the beginning of a

distinct alternative to the truth-table method which was introduced in its simplest form in a previous paper.

17. Professor W. A. Hurwitz: *Topics in the theory of divergent series.*

This paper will appear in full in the January number of this BULLETIN.

18. Dr. Norbert Wiener and Professor F. L. Hitchcock: *A new vector method in integral equations.*

Starting with the Fredholm equation

$$u(x) - \lambda \int_a^b u(y)K(x, y)dy = v(x)$$

the authors say that $f(x)$ is *conjoint* to $g(x)$ if

$$f|g = \int_a^b f(x)g(x)dx - \lambda \int_a^b \int_a^b f(x)g(y)K(x, y)dxdy = 0.$$

They then develop methods of rendering a closed set of functions conjoint by pairs, and they show that if $\{\varphi_n\}$ is a closed conjoint set of functions and

$$\sum_n \varphi_n(x) \frac{\int_a^b v(x)\varphi_n(x)dx}{\varphi_n|\varphi_n}$$

converges uniformly, it represents a solution of the Fredholm equation. Further extensions of this method give series for u that converge in the mean, and enable the determination of the characteristic numbers.

The homogeneous equation

$$u(x) - \lambda \int_a^b u(y)K(x, y)dy = 0$$

is solved by the determination of a function orthogonal to every

$$f(x) - \lambda \int_a^b f(y)K(y, x)dy.$$

A similar method is developed for the solution of the non-homogeneous equation.

All the methods of the present paper are immediate generalizations of vector methods.

19. Dr. Jesse Douglas: *On a certain type of system of* ∞^2 *curves.*

The ∞^2 straight lines of the plane form a family in which the sum of the angles of every triangle is equal to two right angles. The most general families of ∞^2 curves having the same property are the loxodromes, that is those formed of the totality of isogonal trajectories of a given simply infinite system of curves. These are characterized by differential equations of the form

$$y'' = (A + By')(1 + y'^2),$$

where A and B are any functions of x and y such that

$$A_y - B_x = 0.$$

A natural generalization is to inquire as to the systems of ∞^2 plane curves which are such that in every triangle formed of three curves of the system, the sum of the angles differs from π by an amount proportional to the area. The author bases the analysis on the equations of variation, showing that it is sufficient to restrict consideration to triangles of which two sides are formed of infinitesimally adjacent curves, while the third is finitely divergent from these two. The result is that the differential equation of the family must be of the form

$$y'' = (A + By')(1 + y'^2),$$

with $A_y - B_x = k$, where k is the constant of proportionality. A synthetic construction for curve systems of the type so defined is obtained by starting with any system of ∞^1 curves, and drawing trajectories to cut them, not under constant angle, but so that the angle increases by k times the element of area ydx under the trajectory.

The above considerations are easily extended to systems of curves on a general surface.

20. Dr. Jesse Douglas: *Concerning Laguerre's inversion.*

In a previous paper (this BULLETIN, July, 1920) the author made use of a certain representation of the (directed) lines of the plane on the cylinder $x^2 + y^2 = 1$; namely, the line of Hessian coordinates (ω, p) corresponding to the point $(x, y, z) = (\cos\omega, \sin\omega, p)$. The aspect of this representation fundamental for the present paper is that the (oriented)

circles of the plane correspond to the plane sections of the cylinder. By a projective transformation of space the cylinder may be replaced by a general quadric cone. It results that the line transformations in the plane that convert circles into circles form a group, of seven parameters, of which the geometry is identical with projective geometry on a quadric cone.

The last statement is dual to the fact that the point transformations preserving cocircularity form a six-parameter group of which the geometry is identical with projective geometry on a sphere (or any ellipsoid), the identity between the two geometries being based on stereographic projection, which represents the circles of the plane as plane sections of the sphere. The representation described at the beginning thus appears as the dual of stereographic projection.

In the above G_6 of point transformations are included the ordinary inversions, which may be distinguished as corresponding to the perspectives of the sphere. On inquiring what corresponds in the G_1 of line transformations to the perspectives of the cylinder, one comes precisely upon a certain geometric construction described by Laguerre in a paper of 1882 (*Oeuvres*, vol. 2, p. 611). Laguerre studies the properties of his transformation from various points of view, bringing out especially the duality to ordinary inversion, but the aspects above developed appear to be new.

21. Professor J. R. Kline: *Closed connected point sets which are disconnected by the omission of a finite number of points.*

Suppose M is a closed connected point set such that if P is any point of M,

$$M - P = M_1 + M_2,$$

where M_1 and M_2 are two mutually exclusive sets neither of which contains a limit point of the other one. The author shows that M is a continuous curve in the sense of R. L. Moore, i.e. it is closed connected, and connected im kleinen. Suppose G is a closed connected set such that (1) if P is any point in G, $G - P$ is connected, (2) if P_1 and P_2 are any two distinct points of G, we have

$$G - P_1 - P_2 = G_1 + G_2,$$

where G_1 and G_2 are two mutually exclusive sets neither of which contains a limit point of the other one. It is shown that G is a simple closed curve.

22. Professor I. J. Schwatt: *The sum of a series as the solution of a differential equation.*

Boole (*A Treatise on Differential Equations*, 3d edition, pp. 441–450) considers the summation of series that are the solutions of differential equations of the special form

$$\sum_{k=0}^{n} f_k(D)e^{k\theta}u = 0.$$

In this paper a general method for the summation of series as the solutions of linear differential equations is developed and methods are given for solving the equations.

23. Professor I. J. Schwatt: *Method for the summation of a general case of a deranged series.*

Dirichlet, Riemann, Pringsheim and Scheibner have shown that if the terms of a convergent series are deranged, the sum of the resulting series is, in general, different from the sum of the given series. The author has treated two particular cases of deranged series (ARCHIV DER MATHEMATIK UND PHYSIK, vol. 24 (1915), p. 139 and GIORNALE DI MATHEMATICHE, vol. 54 (1916)) and considers in this paper the sum of a general case of a deranged series.

24. Professor I. J. Schwatt: *Higher derivatives of functions of functions.*

Several methods for finding the higher derivatives of functions of functions have been given but they are not in a form convenient for the purposes of application. To this seems to be due the fact that even the leading treatises on the calculus give the higher derivatives of only the simplest functions of functions, in most cases derived by special devices or by induction. Also in the expansion of functions these authors and others find as a rule by actual differentiation the first few derivatives and correspondingly obtain the first few terms of the expansion. In the course of this paper specific references are given. The author has developed methods which enable him to find the general higher derivative of functions of functions and their complete expansions.

25. Professor A. B. Coble: *A covariant of three circles.*

This paper appears in full in the present number of this BULLETIN.

R. G. RICHARDSON,
Secretary.

THE EASTER MEETING OF THE SOCIETY AT CHICAGO.

THE forty-seventh regular meeting of the Chicago Section, constituting the sixteenth regular Western meeting of the Society, was held at the University of Chicago on Friday and Saturday, March 25 and 26, 1921, the first session opening at 10 A.M. on Friday in Room 32, Ryerson Physical Laboratory.

Over sixty persons were present at this meeting, among whom were the following fifty-three members of the Society: Miss Mary C. Ball, Professor G. A. Bliss, Professor Henry Blumberg, Professor R. L. Borger, Dr. H. R. Brahana, Professor C. C. Camp, Professor R. D. Carmichael, Professor E. W. Chittenden, Dr. H. B. Curtis, Professor W. W. Denton, Professor L. E. Dickson, Professor Arnold Dresden, Professor John Eiesland, Professor Arnold Emch, Professor W. B. Ford, Dr. Gladys Gibbens, Dr. Josephine B. Glasgow, Professor W. L. Hart, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor T. F. Holgate, Professor Dunham Jackson, Miss Claribel Kendall, Professor A. M. Kenyon, Professor W. C. Krathwohl, Professor Kurt Laves, Mrs. Mayme I. Logsdon, Professor A. C. Lunn, Professor W. D. MacMillan, Professor T. E. Mason, Professor E. D. Meacham, Professor W. L. Miser, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Dr. C. A. Nelson, Professor C. I. Palmer, Professor Anna H. Palmié, Professor H. L. Rietz, Mr. Irwin Roman, Professor J. B. Shaw, Professor H. E. Slaught, Dr. L. L. Steimley, Mr. E. L. Thompson, Professor E. J. Townsend, Professor E. B. Van Vleck, Professor G. E. Wahlin, Professor E. J. Wilczynski, Professor C. E. Wilder, Professor R. E. Wilson, Professor C. H. Yeaton, Professor A. E. Young, Professor J. W. A. Young.

Forty-seven persons attended the dinner held at the Quadrangle Club on Friday evening.

At the session of Friday afternoon, presided over by President Bliss, Professor Dunham Jackson read an expository paper on *The general theory of approximation by polynomials and trigonometric sums*. This paper appears in the present number of this BULLETIN. At the close of the discussion, which followed the presentation of this paper, Professor

Wilczynski moved and the Section adopted unanimously a resolution expressing to Professor Jackson their appreciation of the admirable way in which he had presented his subject. In view of the great value of the expository work done by Professor de la Vallée Poussin in the field discussed by Professor Jackson, an address expressing to Professor de la Vallée Poussin their appreciation of his work was signed by those present at the dinner on Friday evening.

The other sessions were presided over by Professor Carmichael, Chairman of the Section, relieved for part of Saturday forenoon by Vice-President Jackson.

The papers presented at the meeting are stated below. Professor Schwatt's papers were communicated to the Society by Professor Dunham Jackson. The papers of Professors Miller and Lane were read by title.

1. Professor I. J. Schwatt: *On the expansion of powers of trigonometric functions.*

No satisfactory expressions for the expansions of powers of trigonometric functions seem to have been given. Only two articles on the subject have been found. One, by Dr. Ely in the AMERICAN JOURNAL OF MATHEMATICS, vol. 5 (1882), p. 359, treats only of the expansion of $\sec^p x$ for odd powers of p and obtains by induction a result which involves Euler's numbers. The other, by Shovelton, in the QUARTERLY JOURNAL OF MATHEMATICS, vol. 46 (1915), pp. 220–247, uses the theory of finite differences.

The author has obtained expansions which are believed to be simple and new.

2. Professor I. J. Schwatt: *On the summation of a trigonometric power series.*

To obtain the sum

$$S \equiv \sum_{n=0}^{\infty} n^q \frac{\sin^p (a + ng)}{(b + nh)!} r^n,$$

the author applies the operator $(r\, d/dr)^q$ to S_1, where $(r\, d/dr)^q$ stands for the repetition q times of the operator rd/dr, r and d/dr not being permutable, and where S_1 is of the same form as S except that n^q is wanting. Next S_1 is reduced to

$$S_2 = \sum_{n=0}^{\infty} \frac{\rho^n}{(b + nh)!},$$

where $\rho = re^{(p-2k)\pi i}$ $(k = 0, 1, 2, \cdots, p)$; and an expression for S_2 is obtained by introducing a function $f(n)$ defined by the formula

$$f(n) = \frac{x^n}{(b+n)!}\sum_{k=1}^{h}\theta_k{}^n,$$

where θ_k is one of the kth roots of unity.

3. Professor W. B. Ford: *A disputed point regarding the nature of the continuum.*

In this paper the author considers the question, originally proposed by Du Bois-Reymond, as to whether a decimal whose digits are chosen in an arbitrary manner (as by the throwing of a die) can properly be regarded as a real number. The question forms but a special example of various questions of broader significance which were extensively discussed by Borel, Hadamard, König and others during the decade preceding the war, their conclusions, however, being widely at variance. It is merely desired in the present paper to point out that such numbers must be regarded as acceptable in case one is to regard the familiar cardinal number theory of Cantor, particularly as regards the continuum, as having a proper place in mathematics, since the derivation of Cantor's results clearly depends upon the use of decimals whose digits are assigned entirely at random.

4. Mrs. Mayme I. Logsdon: *The equivalence of pairs of hermitian forms.*

In this paper the author makes a generalization of the theory of elementary divisors for hermitian λ-matrices with special application to pairs of hermitian forms. If two hermitian matrices A and B are equivalent in the sense that there exist non-singular matrices C and D with determinants free of λ such that $B = CAD$, it is shown that there exists a non-singular matrix P with determinant free of λ such that $B = \overline{P}'AP$, where $\overline{P}$ means the matrix formed from P by replacing each element by its conjugate imaginary and P' means P transposed. It is found that in the case of hermitian forms as well as bilinear and quadratic forms the coincidence of the elementary divisors is a necessary and sufficient condition for equivalence.

The Weierstrass reduction is found to hold where one of the forms is definite, thereby insuring reality of all the elementary

divisors. When complex elementary divisors are present they enter in conjugate pairs. In this case it was found possible to regularize the matrix with respect to the two conjugate linear factors simultaneously and to expand simultaneously with respect to these two factors the terms representing the adjoint form. A simple canonical form was obtained.

5. Mr. C. C. MacDuffee (introduced by Professor L. E. Dickson): *Invariants and vector covariants of linear algebras without the associative law.*

A vector covariant of a linear algebra has been defined by Professor Hazlett (TRANSACTIONS, vol. 19) as a covariant which involves one or more of the units $e_1, \cdots, e_n$. In this paper the author considers the general linear algebra in which the commutative and associative laws of multiplication are not assumed, and shows that, after a vector covariant of weight μ has been expressed linearly in the units, the coefficients of the units are transformed cogrediently, apart from the factor $D^{-\mu}$, with the coefficients of the general number of the algebra under a linear transformation of determinant D. Therefore the coefficients of a vector covariant may be inserted in place of the variables in the covariants of the characteristic equations to give additional relative invariants. After defining a determinant whose elements are hypercomplex numbers for which multiplication is not assumed to be commutative or associative, and after showing that certain elementary theorems concerning determinants apply to these hypercomplex determinants, it is proved that the hypercomplex determinant each of whose n rows is $e_1, e_2, \cdots, e_n$ is a vector covariant of weight 1 for every manner of grouping the factors in each term.

6. Professor E. J. Wilczynski: *Some projective generalizations of geodesics.*

The two most important properties of geodesics are: (*a*) their connection, as extremal curves, with the length of arc integral; (*b*) the property that the osculating plane at any point on such a curve contains the corresponding surface normal.

In this paper, the author considers the generalizations which arise from these two points of view; (*a*) by replacing the length of arc integral by some other integral invariant of the same

general form; (*b*) by replacing the congruence of normals by some other congruence. The two classes of curves obtained in this way do not coincide; but both are included in a larger class of two-parameter curves defined by a second order differential equation of the form

$$\frac{d^2v}{du^2} = M\left(\frac{dv}{du}\right)^3 + N\left(\frac{dv}{du}\right)^2 + P\frac{dv}{du} + Q,$$

where M, N, P, and Q are functions of u and v, the parameters of a surface point. These classes of curves may be characterized by properties of the osculating planes of the curves of a family which pass through a given surface point and dualistically related properties. It then remains to locate the two more special classes mentioned above within the larger class.

7. Professor W. L. Hart: *Summable infinite determinants.*

It has been customary to define the value of an infinite determinant (1) $D = |a_{ij}|$ $(i, j = 1, 2, \cdots)$ by the equation $D = \lim_{n=\infty} D_n$, in case the limit exists, where D_n is the determinant of order n formed by the elements of D for $i, j = 1, 2, \cdots, n$. Utilizing Cesàro summability of order r, the author assigns the value D to the determinant (1) if the sequence $(D_n; n = 1, 2. \cdots)$ is summable to the value D. The results of the paper are restricted to the case $r = 1$. If in (1) we choose all elements as zero except for $a_{2k-1,\,2k} = 1$ and $a_{2k,\,2k-1} = -1$ $(k = 1, 2, \cdots)$, D is summable to the value $\frac{1}{2}$. If quantities c_{ij} are added to the elements of this determinant, then, in case $\Sigma|c_{ij}|$ converges, the new determinant thus obtained is summable and possesses most of the useful properties of normal infinite determinants. There is also exhibited a more complicated type of summable determinants, related in a simple way to a determinant (1), summable to the rational value p/q $(p < q)$, all of whose elements are either $+1$, -1, or 0. It should be noted that no cases in the solution of infinite systems of linear equations can be solved by means of summable determinants which are not also solvable by means of normal infinite determinants.

8. Professor Henry Blumberg: *New properties of all functions.*

For the sake of simplicity the following statements will be made to refer to functions $z = f(x, y)$ of two variables. A point (ξ, η, ζ), where $\zeta = f(\xi, \eta)$, of the surface $z = f(x, y)$

is said to be *densely approached* if for every $\gamma > 0$ two numbers $\alpha > 0$ and $\beta > 0$ exist such that the projections upon the XY-plane of the surface points in the rectangular parallelopiped $\xi - \alpha \leqq x \leqq \xi + \alpha$, $\eta - \beta \leqq y \leqq \eta + \beta$, $\zeta - \gamma \leqq z \leqq \zeta + \gamma$ are everywhere dense in the rectangle $\xi - \alpha \leqq x \leqq \xi + \alpha$, $\eta - \beta \leqq y \leqq \eta + \beta$. The author has found that the points (x, y) for which (x, y, z) is densely approached constitute a residual set (set of second category, according to Baire) of the XY-plane; conversely, for every residual set R, there exists a function $f(x, y)$ such that (x, y, z) is densely approached if and only if (x, y) is in R. It follows that for every function $f(x, y)$ a dense set D of the XY-plane exists, such that f is continuous if it is defined only in D. The results flow from a general theorem concerning *closed region-point* relations. There are generalizations for n dimensions, function space and more general spaces.

9. Professor E. B. Van Vleck: *On non-loxodromic substitution groups in n dimensions.*

This paper is a continuation of a previous paper. For linear substitutions $z' = (az + b)/(cz + d)$ without a common pole there exist two types of non-loxodromic groups, the one consisting of non-loxodromic substitutions having a common invariant circle, while the other contains exclusively elliptic substitutions which are without a common invariant circle. On the author's definition of a non-loxodromic substitution in n dimensions corresponding results hold, but the analog of the elliptic group exists only in space of an odd number of dimensions.

10. Professor G. A. Miller: *An overlooked infinite system of groups of order pq^2, p and q being prime numbers.*

Lists of the possible abstract groups of order pq^2, p and q being prime numbers, were published almost simultaneously by Cole and Glover in the AMERICAN JOURNAL OF MATHEMATICS, vol. 15 (1893), and by O. Hölder in the MATHEMATISCHE ANNALEN, vol. 43 (1893). In a subsequent article published in volume 46 of the latter journal, Hölder directed attention to the fact that the enumeration of these groups contained in the first article mentioned above was incomplete. In the present article the author directs attention to the fact that Hölder's enumeration is also incomplete and observes

that the same incompleteness appears in the enumerations found in the first and also in the second edition of W. Burnside's *Theory of Groups of Finite Order* (1897 and 1911), as well as elsewhere.

There is a system which contains $q - 1$ distinct abstract groups for a given value of p and q, while only one of these groups is found in the published lists.

11. Professor L. E. Dickson: *Fallacies and misconceptions in diophantine analysis.*

This paper has appeared in the April number of this BULLETIN.

12. Professor L. E. Dickson: *A new method in diophantine analysis.*

This paper has appeared in the May number of this BULLETIN.

13. Professor T. H. Hildebrandt: *On a general theory of functions.* (Preliminary report.)

In recent years there have been a number of attempts in the direction of extending the results of the theory of functions of n variables to other cases, that of infinitely many variables, and functionals of continuous functions having received the most attention. This work has been done chiefly by Fréchet, Hart and Gateaux. In this paper an attempt is made to extend by generalization the domain of application of the results already attained, and derive new ones in the general domain. The author uses the basis recently suggested by Lamson (AMERICAN JOURNAL OF MATHEMATICS, vol. 42 (1920), p. 245). So far there have been obtained theorems of mean value, a theorem on implicit functions, existence theorems of differential equations, and some theorems on analytic functions.

14. Professor Arnold Dresden: *Some new formulas in combinatory analysis.*

In this paper the author presents some formulas involving binomial coefficients, which proved of importance in work on symmetric functions with which he is occupied. A search of the literature of combinatory analysis has failed to bring these formulas to light. They are therefore believed to be new.

15. Professor J. B. Shaw: *Generational definition of linear associative hypernumbers.*

Let the n^2 hypernumbers λ_{ij} $(i, j = 1, \cdots, n)$ and certain nilpotent hypernumbers θ_{rs} be given; then the general linear associative hypernumber is of the form

$$\Sigma\lambda_{ij}A_{ij}(\theta_{aj})B_{ijk}(\theta_{bk})C_{ijkl}(\theta_{cl}) \cdots.$$

The laws by which the λ's and the θ's may be permuted in a product are discussed in this paper. They are essential to the algebra to which the general hypernumber belongs.

16. Professor J. B. Shaw: *On Hamiltonian products. Second paper.**

Our range is the totality of functions of a single real variable such that the product of any two is integrable (Lebesgue, Stieltjes, etc.) over a given interval. Each function is represented with a different argument save as specified. Let $\alpha_1(s_1), \alpha_2(s_2), \cdots$ be functions of the totality considered. Then there are two processes considered to be fundamental. The first consists in constructing out of n functions a function of n arguments as follows. Write the product of the n functions $\alpha_1(s_1)\alpha_2(s_2)\cdots\alpha_n(s_n)$. Form similar products from this by permuting the arguments only into every possible arrangement, giving the result a sign $+$ or $-$ according to the inversions, as in determinants. The results are then added. This is represented by $A_n\alpha_1\alpha_2\cdots\alpha_n$.

The second fundamental process consists in starting from the product $\alpha_1(s_1)\alpha_2(s_2)\cdots\alpha_n(s_n)$ as before, but now forming products by making two arguments identical in every possible manner, the sign of the term being determined by the number of inversions necessary to bring the two functions containing the identical arguments together. Each term is then integrated as to the equal arguments over the given interval. If we add the results, we obtain a function of $n - 2$ arguments indicated by $A_{n-2}\alpha_1\alpha_2\cdots\alpha_n$. Repeating the process, or what is the same thing, making two pairs of arguments identical in every possible way, and then integrating as to each of the two, determining signs as before, leads to the function $A_{n-4}\alpha_1\alpha_2\cdots\alpha_n$. Then we define the Hamiltonian product of the functions $\alpha_1, \alpha_2, \cdots, \alpha_n$ as the function

* See this BULLETIN, (2), vol. 16 (1910), p. 304.

$H[\alpha_1(s_1), \alpha_2(s_2), \cdots, \alpha_n(s_n)] = A_n\alpha_1\alpha_2\cdots\alpha_n + A_{n-2}\alpha_1\alpha_2\cdots\alpha_n + \cdots + A_1\alpha_1\alpha_2\cdots\alpha_n$ (or $A_0\alpha_1\alpha_2\cdots\alpha_n$) for the last term. The author considers the resulting theorems.

17. Professor F. E. Wood: *Congruences characterized by certain coincidences.*

The equations of certain lines covariantly related to a general congruence of lines have previously been obtained with respect to various tetrahedra of reference. In this paper the author obtains the equations of these lines with respect to a single fundamental tetrahedron of the congruence as well as the general equations which transform the equations of any configuration related to either of two other tetrahedra of reference into the equations of the same configuration related to this fundamental tetrahedron.

The possible coincidences of these lines, for every line of the congruences, are studied in detail. The congruences characterized by certain coincidences are obtained in a canonical form. In particular, the class of congruences for which the directrix of the first kind of each focal sheet coincides with the directrix of the second kind of the other sheet is found to be the same as the class of congruences associated with functions of a complex variable by Wilczynski.

18. Professor E. P. Lane: *A general theory of congruences.*

Geometers who have considered conjugate nets from the point of view of projective differential geometry have for the most part been content to study a single net, together with its attendant configurations, and have been forced to carry out a tedious transformation whenever it has been desired to consider another conjugate net on the same surface.

The author lays the foundation for a theory of all the conjugate nets on a surface, using the following method. The asymptotic net is taken as parametric. Then the fundamental invariants and covariants of an arbitrary conjugate net are calculated in terms of the asymptotic parameters. The resulting formulas are comparatively simple, and their usefulness is demonstrated by applications to pencils of nets, harmonic nets, isothermally conjugate nets, and plane nets.

Among other theorems, it is proved that if there are more than three harmonic nets in one pencil, the pencil is an isother-

mally conjugate quadratic pencil. And if there are exactly three harmonic nets in a pencil, the pencil is the Segre-Darboux pencil. It is shown that the envelope of the ray of a point for all nets of a pencil is a conic.

19. Professor John Eiesland: *The group of motions of an Einstein space.*

The question to what extent an Einstein space is determined by its group of motions seems to be of interest both from a geometrical and from a physical standpoint. Three assumptions L, M, and N have been made. L is of group-theoretical nature and postulates that the general space $\Sigma_1^4 g_{ik}dx_i dx_k$ admits a certain 4-parameter intransitive group of motions which keeps the origin of the space $t = const.$ fixed. The general space admitting the group G_4 is shown to be of the form

$$(1) \quad -ds^2 = (1+\varphi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) - 2\varphi_3 dR dt - \varphi_1 dt^2,$$

where φ_1, φ_2, φ_3 are functions of R. The static space, for which $\varphi_3 = 0$, or,

$$(2) \quad -ds^2 = (1+\varphi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) - \varphi^1 dt^2,$$

is then considered. Adopting two additional assumptions, M and N, we arrive at Schwarzschild's form. The two assumptions are as follows. M. The discriminant of (2) is invariant and equal to $R^4\sin^2(\theta - c^2)$. (This assumption is also known as Kottler's fundamental hypothesis.*) N. The sum of the principal Riemannian curvatures of the space

$$ds_0^2 = (1+\varphi_2)dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

is equal to zero.

By varying the last assumption other forms of approximately Euclidean spaces have been obtained. Thus, if the sum is inversely proportional to the fourth power of R, we obtain Weyl's line-element for the case of a spherical mass with a constant electrical charge. In the final part of the paper is considered a gravitational space admitting a two-parameter group of motions, namely a translation along the t-axis and a rotation about the z-axis.

ARNOLD DRESDEN,
Secretary of the Chicago Section.

* F. Kottler, *Grundlagen der Einsteinschen Gravitationstheorie*, ANNALEN DER PHYSIK, (4), vol. 56, p. 409.

THE APRIL MEETING OF THE SAN FRANCISCO SECTION.

The thirty-seventh regular meeting of the San Francisco Section of the American Mathematical Society was held at Stanford University, on Saturday, April 9. The chairman of the Section, Professor D. N. Lehmer, presided. The total attendance was twenty-eight, including the following fourteen members of the Society:

Professor R. E. Allardice, Professor B. A. Bernstein, Professor H. F. Blichfeldt, Professor Florian Cajori, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Dr. F. R. Morris, Professor C. A. Noble, Professor T. M. Putnam, Dr. S. E. Urner, Dr. A. R. Williams.

After the regular programme, interesting papers were read by Professor D. L. Webster, of Stanford University, on the quantum theory, and by Dr. T. L. Kelly, also of Stanford University, on the new theory of dispersion.

The dates of the next two regular meetings of the Section were fixed as October 22, 1921, and April 8, 1922.

Titles and abstracts of the papers read at this meeting follow below. Professor Bell's papers were read by title.

1. Professor Florian Cajori: *Euclid of Alexandria and the bust of Euclid of Megara.*

Professor Cajori proves that the portrait issued by the Open Court Publishing Company as representing Euclid of Alexandria, the geometrician, is in fact a portrait of Euclid of Megara. This article appeared in SCIENCE for April 29, 1921.

2. Professor Florian Cajori: *The spread of the Newtonian and the Leibnizian notations of the calculus.*

This paper appears in the present number of this BULLETIN.

3. Professor H. F. Blichfeldt: *The approximate solution in integers of a set of linear equations.*

Consider the n functions $f_1 = |x_1 + \alpha_1 z + a_1|$, $\cdots$, $f_n = |x_n + \alpha_n z + a_n|$, involving $n+1$ variables x_1, $\cdots$, x_n, z and $2n$ given constants α_1, $\cdots$, α_n, a_1, $\cdots$, a_n, the

numbers $\alpha_1, \cdots, \alpha_n$ being irrationals which satisfy no equation of the form $k_1\alpha_1 + \cdots + k_n\alpha_n + k = 0$, where $k, k_1, \cdots, k_n$ are integers not all zero. By a theorem of Kronecker it follows that, if a positive quantity ϵ as small as we please is given, then a set of integers $x_1, \cdots, x_n, z$ exist for which the functions f_i are all $< \epsilon$. It is proved in this paper that, given an arbitrary function $\varphi(z)$ of a positive integer z, provided that it represents a positive number approaching infinity with z and that $\varphi(z+1) \geqq \varphi(z)$, a set of irrationals $\alpha_1, \cdots, \alpha_n$ may be defined, restricted as above, such that the condition $F < 1/\varphi(z)$ is satisfied by only a finite number of sets of integers $x_1, \cdots, x_n, z$; here F represents any one of the elementary symmetric functions of degree not higher than $n-1$ of the n functions f_i. On the other hand, whenever the set of irrationals $\alpha_1, \cdots, \alpha_n$ is given, restricted as above, a set of integers $x_1, \cdots, x_n, z$ exists for which every $f_i < \epsilon$, and at the same time $\Pi_{i=1}^{n} f_i < N/z$, where N depends on n only.

4. Professor M. W. Haskell: *Autopolar curves and surfaces.*

In this paper, the author gives the following method for deriving all autopolar curves and surfaces. There is a (1, 1) correspondence between the line-elements of a given curve and the line-elements of its polar reciprocal with respect to a given conic. Every pair of corresponding line-elements determines a pair of line-elements belonging to the given conic. The locus of the foci of the involution defined by these two pairs of line-elements is a curve which is autopolar with respect to the given conic. Every autopolar curve with respect to the given conic can be derived in this way. Curves and surfaces autopolar with respect to a given quadric surface can be derived in the same way, substituting surface-elements for line elements.

The author then gives illustrations of multiply auto-polar curves: a special quintic with five cusps autopolar with respect to six conics, and a special quintic with three cusps and three conjugate points autopolar with respect to four conics.

5. Mr. P. H. Daus (introduced by Professor Lehmer): *Normal ternary continued fractions.*

The author discusses an extension of Jacobi's ternary continued fraction algorithm, and points out certain similarities between it and the ordinary continued fraction expansion.

6. Mr. D. V. Steed (introduced by Professor Lehmer): *The hyperspace generalization of the lines on the cubic surface.*

In this paper it is shown that a necessary condition for the existence of a finite number of linear spaces of dimensions d on the general hypersurface of order r in space of n dimensions is that r, n, and d satisfy the equation

$$\binom{r+d}{d}=(n-d)(d+1).$$

In particular the case where $d = 1$ is considered, and a method for the enumeration of the lines on the general hypersurface of order $2n - 3$ in space of n dimensions is developed. The numbers found for space of dimensions four, five, six, and seven are 2875; 698,005; 306,142,821; and 211,039,426,895, respectively, but by means of the recursion formulas given it is possible to make the corresponding enumeration for space of any dimension.

7. Professor D. N. Lehmer: *On the computation of interest on certain kinds of investments.*

The equations of high degree which are often involved in the computation of the rate of interest on annuities, when treated by Newton's method, yield certain correction formulas which avail to compute the rate to as high a degree of accuracy as may be desired and which require of the computer no more knowledge of the theory of equations or of algebra than is necessary to substitute quantities in a formula. The author exhibits these correction formulas for the rate of interest on a bond, bought at a certain price, and on an annuity whose present value, or whose amount, is given.

The paper is to appear in the AMERICAN JOURNAL OF ACCOUNTANCY.

8. Professor E. T. Bell: *The Bernoullian functions occurring in the arithmetical applications of elliptic functions.*

In applying elliptic functions to theorems on divisors, Glaisher, Halphen and others have given formulas involving Bernoullian functions.

In this paper it is shown that sixteen distinct Bernoullian functions, and no more, can arise in the arithmetic applications of elliptic theta functions and their quotients. The set is considered in detail with applications.

9. Professor E. T. Bell: *Anharmonic polynomial generalizations of the numbers of Bernoulli and Euler.*

In this paper the author discusses two cyclic sets and one anharmonic set which for every degree n degenerate when $z = 1$ either to Bernoulli or Euler numbers of rank n or to others naturally dependent upon these. The entire subject is developed by means of an extension of the symbolic calculus of Blissard and Lucas, and is shown to be simply isomorphic to the algebra of the twelve Jacobian elliptic functions of Glaisher. This analysis leads to several novel features, among which may be mentioned: the definition and solution of purely symbolic linear difference equations; symbolic addition theorems whereby the polynomials of degree $m + n$ can be easily calculated from those of degrees m, n; and a wholly new interpretation of Kronecker's symmetric functions which he took as the point of departure for his discussion of Bernoulli numbers. It is shown that the polynomials cannot be obtained by linear recurrences, and the appropriate recurrences are derived. Finally congruences to a prime modulus are discussed in detail. The algebraic relations and congruences between the polynomials degenerate for $z = 1$ to recurrences and congruences for Bernoulli and Euler numbers. Some of the degenerate cases were given by Lucas and others, thus affording checks.

10. Professor E. T. Bell: *Note on the prime divisors of the numerators of Bernoulli's numbers.*

This note, which will appear shortly in the AMERICAN MATHEMATICAL MONTHLY, contains a generalization of a result due to John Couch Adams, viz. if p is an odd prime which divides neither $2^r + 1$ nor $2^r - 1$, then the numerator of B_{2pr} (in Lucas' notation) is divisible by p. For $r = 1$ we get Adams' theorem. The proof follows from the series for sn u.

11. Professor E. T. Bell: *Proof of an arithmetic theorem due to Liouville.*

This paper appeared in the March number of this BULLETIN.

12. Professor E. T. Bell: *On a general arithmetic formula of Liouville.*

This paper appeared in the April number of this BULLETIN.

B. A. BERNSTEIN,
Secretary of the Section.

THE GENERAL THEORY OF APPROXIMATION BY POLYNOMIALS AND TRIGONOMETRIC SUMS.

REPORT PRESENTED BEFORE THE AMERICAN MATHEMATICAL SOCIETY AT THE SYMPOSIUM IN CHICAGO ON MARCH 26, 1921.

BY PROFESSOR DUNHAM JACKSON.

1. *Introduction.* A great part of the progress made in mathematical analysis during the last hundred years has been closely related, either logically or historically, to the study of Taylor's and Fourier's series. The power series is fundamental, or can be made fundamental, for the theory of functions of complex variables, and the theory of functions of real variables has been elevated to its present dignity and scope largely by the successive additions made to meet the demands of the inquirer into the properties of trigonometric series.*

One of the outlying portions of the structure is built around the problem of the approximate representation of an arbitrary function by means of polynomials or by means of finite trigonometric sums in general; that is, with the admission of coefficients other than those of a specified number of terms of a particular series. This theory, which has grown to its present extent mainly within twenty years, forms the subject of the following report. It can not be set off from the rest by any sharp line of demarcation, but there are certain well-marked processes of development which it is not difficult to trace. It has seemed expedient for various reasons to make this paper an introduction to the literature of the subject, rather than an independent exposition of any considerable part of the theory. Even with this limitation, the treatment is merely illustrative, not in any sense exhaustive.

2. *Tchebychef's Theory and its Generalizations.* Between 1850 and 1860, Tchebychef† discussed the problem of de-

* Cf. Van Vleck, *The influence of Fourier's series upon the development of mathematics*, SCIENCE, vol. 39 (1914), pp. 113–124.

† Tchebychef 1 (this form of reference will be adopted when there is occasion to cite more than one paper by the same author), *Théorie des mécanismes connus sous le nom de parallélogrammes*, MÉMOIRES PRÉSENTÉS À L'ACADÉMIE IMPÉRIALE DES SCIENCES DE ST.-PÉTERSBOURG PAR DIVERS SAVANTS, vol. 7 (1854), pp. 539–568; *Oeuvres*, vol. 1, Petrograd, 1899, pp. 111–143.

Tchebychef 2, *Sur les questions de minima qui se rattachent à la représenta-*

termining a polynomial $P_n(x)$, of given degree n, to approximate a given continuous function $f(x)$, in such a way that the maximum of the absolute value of the error,

$$\text{Max. } |f(x) - P_n(x)|,$$

shall be as small as possible. His reasoning, as recorded at that early date, was naturally incomplete, according to present standards. It was put into modern form by Kirchberger* in 1902. The idea was then rapidly carried further. It was applied by Fréchet† and by J. W. Young‡ to problems of a considerably higher degree of generality, and in particular to representation by finite trigonometric sums, by Tonelli§ to functions of more than one variable (a phase already touched upon by Kirchberger and Fréchet), and by Sibirani‖ to representation by linear combinations of a given set of linearly independent functions generally; and it has been extended in a variety of other ways.

The most striking facts with regard to the Tchebychef polynomial of given degree n, for a given continuous function $f(x)$, in a given interval $a \leqq x \leqq b$, are that it is uniquely determined, and that the error $f(x) - P_n(x)$ takes on its greatest numerical value not just once, but at least $n + 2$ times, alternately with opposite signs.¶ When $n = 0$, for

tion approximative des fonctions, MÉMOIRES DE L'ACADÉMIE IMPÉRIALE DES SCIENCES DE ST.-PETERSBOURG, (6), SCIENCES MATHÉMATIQUES ET PHYSIQUES, vol. 7, (1859), pp. 199–291; *Oeuvres*, vol. 1, pp. 273–378.

In preparing the present paper, I have consulted only the collected works, from which the citations of the original memoirs are quoted.

* Kirchberger, *Ueber Tchebychefsche Annäherungsmethoden*, Dissertation, Göttingen, 1902. See also Blichfeldt, *Note on the functions of the form* $f(x) \equiv \varphi(x) + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_n$ *which in a given interval differ the least possible from zero*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 2 (1901), pp. 100–102.

† Fréchet 1, *Sur l'approximation des fonctions par des suites trigonométriques limitées*, COMPTES RENDUS, vol. 144 (1907), pp. 124–125; Fréchet 2, *Sur l'approximation des fonctions continues périodiques par les sommes trigonométriques limitées*, ANNALES DE L'ECOLE NORMALE SUPÉRIEURE, (3), vol. 25 (1908), pp. 43–56.

‡ J. W. Young, *General theory of approximation by functions involving a given number of arbitrary parameters*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 8 (1907), pp. 331–344.

§ Tonelli, *I polinomi d'approssimazione di Tchebychev*, ANNALI DI MATEMATICA, (3), vol. 15 (1908), pp. 47–119.

‖ Sibirani, *Sulla rappresentazione approssimata delle funzioni*, ANNALI DI MATEMATICA, (3), vol. 16 (1909), pp. 203–221.

¶ In the paper of 1854, the earliest reference to the subject with which I am acquainted, Tchebychef says (*Oeuvres*, vol. 1, p. 114):

"Soit fx une fonction donnée, U un polynome du degré n avec des coefficients arbitraires. Si l'on choisit ces coefficients de manière à ce que la

example, it is evident that the best constant is that midway between the greatest and least values of $f(x)$, so that the maximum deviation is reached at least twice, once positively and once negatively. A little experimentation with a ruler and a curve drawn on paper will leave little doubt as to the correctness of the statement for $n = 1$, and perhaps will suggest at the same time the idea of the general proof. The latter, however, is by no means trivial, and it is a very satisfactory exercise in the application of elementary theorems of algebra and of the simplest principles of analysis.

For the corresponding trigonometric problem, let it be supposed that $f(x)$ is of period 2π, and is continuous for all real values of x. Among all expressions of the form

$$T_n(x) = a_0 + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx,$$

that is, among all finite trigonometric sums of order n, there will be one and just one for which the maximum of the quantity $|f(x) - T_n(x)|$ has the smallest possible value. This particular sum is characterized by the fact that the quantity $f(x) - T_n(x)$ reaches its greatest numerical value at least $2n + 2$ times in a period, alternately with opposite signs. It will be seen that the number $2n + 2$ here, like the number $n + 2$ in the polynomial case, is one more than the number of arbitrary coefficients in question.

Except for certain specific references in a later section, there is perhaps no occasion to dwell upon the subject longer here, since its main features are very readably presented in standard treatises.* The explicit determination of the approximating function, or of the degree of approximation attained, is extraordinarily difficult, even in relatively simple cases, because the dependence of the approximating function on $f(x)$ is not linear. It is especially noteworthy that the Tchebychef

différence $fx - U$, depuis $x = a - h$, jusqu'à $x = a + h$, reste dans les limites les plus rapprochées de 0, la différence $fx - U$ jouira, *comme on le sait*, de cette propriété:

'Parmi les valeurs les plus grandes et les plus petites de la différence $fx - U$ entre les limites $x = a - h$, $x = a + h$, on trouve au moins $n + 2$ fois la même valeur numérique.'"

The italics are mine. Kirchberger, op. cit., p. 6, states that the problem was originally proposed by Poncelet.

* Cf., e.g., Borel, *Leçons sur les Fonctions de Variables Réelles et les Développements en Séries de Polynomes*, Paris, 1905, pp. 82–92; de la Vallée Poussin 1, *Leçons sur l'Approximation des Fonctions d'une Variable Réelle*, Paris, 1919, see chapters 6, 7.

theory did not lead in any direct way to a recognition of the fundamental fact brought out in the next section, that there exist uniformly convergent processes of approximation by polynomials or by finite trigonometric sums, for an arbitrary continuous function $f(x)$, although the approximating functions of Tchebychef by definition yield the most rapidly convergent of all such processes.

3. *Weierstrass's Theorem.* Let $f(x)$ be an arbitrary continuous function over the interval $a \leqq x \leqq b$, and let ϵ be an arbitrary positive quantity. Then there will always exist a polynomial $P(x)$ such that $|f(x) - P(x)| < \epsilon$ throughout the interval. This theorem was first published by Weierstrass* in 1885. Almost at the same time, and independently of Weierstrass, Runge,† without formulating this particular conclusion, supplied the materials for a second proof, quite different in form. Other writers successively added other demonstrations, in great variety.‡ A time came when there was no longer any distinction in inventing a proof of Weierstrass's theorem, unless the new method could be shown to possess some specific excellence, in the way of simplicity, for example, or in rapidity of convergence. The question of degree of convergence will be the principal concern of the remainder of the paper. In respect to simplicity, mention may be made particularly of the proof of Lebesgue,§ which depends on the application of the binomial theorem to the representation of a function whose graph is a broken line, the proof of Landau,‖ which makes use of an especially con-

* Weierstrass 1, *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen*, BERLINER SITZUNGSBERICHTE, 1885, pp. 633–639.

† Runge, *Über die Darstellung willkürlicher Functionen*, ACTA MATHEMATICA, vol. 7 (1885), pp. 387–392, together with an earlier paper by the same author: *Zur Theorie der eindeutigen analytischen Functionen*, ACTA MATHEMATICA, vol. 6 (1885), pp. 229–244; pp. 236–237. The first-named paper deals with the approximation of an arbitrary continuous function by means of a rational function, while the other supplies the necessary facts about the approximate representation of rational functions by means of polynomials.

‡ Cf., e.g., Borel, op. cit., pp. 50–61.

§ Lebesgue 1, *Sur l'approximation des fonctions*, BULLETIN DES SCIENCES MATHÉMATIQUES, (2), vol. 22 (1898), pp. 278–287; cf. de la Vallée Poussin 1, pp. 3–5.

‖ Landau, *Über die Approximation einer stetigen Funktion durch eine ganze rationale Funktion*, RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO, vol. 25 (1908), pp. 337–345.

venient form of (so-called *singular*) definite integral, and Simon's modification* of Landau's proof, in which the definite integral is replaced by a finite sum.

The trigonometric form of the theorem, to the effect that an arbitrary continuous function of period 2π can be uniformly represented by a finite trigonometric sum with any assigned degree of accuracy, was established by Weierstrass himself, not in the first paper already cited, but immediately afterwards.† The polynomial and trigonometric cases are not only susceptible of parallel treatment, but are readily convertible one into the other by a simple change of variable. Among the direct proofs for the trigonometric case may be mentioned, for simplicity and elegance, those of de la Vallée Poussin,‡ using a definite integral, and Kryloff,§ using the finite sum which corresponds to de la Vallée Poussin's integral.

4. *First Studies of Degree of Convergence. De la Vallée Poussin's Problem.* It has long been known, in more or less detail, that there is a relation between the properties of continuity of a function and the degree of accuracy that can be attained in its approximate representation by specified means. For example, Picard,‖ in his *Traité d'Analyse*, points out incidentally that if $f(x)$ is a function of period 2π possessing a kth derivative which is (essentially) of limited variation, the coefficient of $\cos nx$ or $\sin nx$ in the Fourier series for $f(x)$ does not exceed a constant multiple of $1/n^{k+1}$ in absolute value. This leads to a theorem about the degree of approximation to $f(x)$ given by the partial sum of its Fourier series, that is by a particular trigonometric sum of the nth order.

Lebesgue,¶ in 1908, formally proposed the problem of discussing the relation between the accuracy of polynomial ap-

*Simon, *A formula of polynomial interpolation*, ANNALS OF MATHEMATICS, (2), vol. 19 (1918), pp. 242–245.

† Weierstrass 2, same title as his first paper, BERLINER SITZUNGSBERICHTE, 1885, pp. 789–805.

‡ de la Vallée Poussin 2, *Sur l'approximation des fonctions d'une variable réelle et de leurs dérivées par des polynômes et des suites limitées de Fourier*, BULLETINS DE L'ACADÉMIE ROYALE DE BELGIQUE, Classe des Sciences, 1908, pp. 193–254.

§ Kryloff, *Sur quelques formules d'interpolation généralisée*, BULLETIN DES SCIENCES MATHÉMATIQUES, (2), vol. 41 (1917), pp. 309–320.

‖ Picard, *Traité d'Analyse*, 2nd ed., vol. 1, pp. 252–253, 255–256.

¶ Lebesgue 2, *Sur la représentation approchée des fonctions*, RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO, vol. 26 (1908), pp. 325–328.

proximation and the requisite degree of the polynomial, and indicated that if $f(x)$ satisfies a Lipschitz condition for $a \leqq x \leqq b$, it can be approximately represented by a polynomial of the nth degree with an error not exceeding a constant multiple of $\sqrt{(\log n)/n}$. This result was improved by de la Vallée Poussin,* who obtained the limit $1/\sqrt{n}$ in place of that just mentioned.† For a more restricted class of functions, retaining enough generality to include any function whose graph is a broken line with a finite number of segments, he found‡ the still closer limit $1/n$. Then he added,§

"Il serait très intéressant de savoir s'il est impossible de représenter l'ordonnée d'une ligne polygonale avec une approximation d'ordre supérieur à $1:n$ par un polynôme de degré n."

This remark has been the direct or indirect occasion of most of the subsequent work on the subject.

5. *Inner Limit of Approximation. S. Bernstein's Theory.* The simplest example of a function coming within the specifications of the problem is the function $|x|$, considered in the interval from -1 to 1. It was shown by de la Vallée Poussin‖ that the maximum error of an approximating polynomial of degree n can not approach zero faster than $1/(n \log^3 n)$. S. Bernstein¶ and the writer** independently replaced this limit by $1/(n \log n)$. The final solution, verifying de la Vallée Poussin's surmise that $1/n$ is the actual limit, was given by S. Bernstein,†† in a notable prize essay for the Belgian Academy,

* de la Vallée Poussin 2, already cited, p. 222.

† The hypothesis was somewhat differently stated by de la Vallée Poussin, but his analysis is immediately applicable to the case of a Lipschitz condition; cf. D. Jackson 1, cited below, p. 10, footnote.

‡ de la Vallée Poussin 3, *Note sur l'approximation par un polynôme d'une fonction dont la dérivée est à variation bornée*, BULLETINS DE L'ACADÉMIE ROYALE DE BELGIQUE, Classe des Sciences, 1908, pp. 403–410.

§Footnote, p. 403.

‖ de la Vallée Poussin 4, *Sur les polynomes d'approximation et la représentation approchée d'un angle*, BULLETINS DE L'ACADÉMIE ROYALE DE BELGIQUE, Classe des Sciences, 1910, pp. 808–844.

¶ S. Bernstein 1, *Sur l'approximation des fonctions continues par des polynomes*, COMPTES RENDUS, vol. 152 (1911), pp. 502–504.

** D. Jackson 1, *Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung*, Dissertation, Göttingen, 1911; see pp. 49–52.

†† S. Bernstein 2, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné*, Mémoire couronné, Brussels, 1912. (Included in MÉMOIRES PUBLIÉS PAR LA CLASSE DES SCIENCES DE

with de la Vallée Poussin's problem for its text. Expanding into a discussion of a wide range of questions in the theory of polynomial approximation generally, the essay contains a number of results comparable in interest with the accomplishment of its original purpose. One of these, by reason of its simplicity and its far-reaching consequences, may well be regarded as one of the most remarkable theorems of recent times. It will be worth while to dwell on it at some length. Here, again, it is possible to speak either of polynomials or of trigonometric sums. The statement is more striking in terms of the latter. For the trigonometric case, the theorem is as follows:

Let $T_n(x)$ be an arbitrary trigonometric sum of order n, and let L be the maximum of its absolute value. Then the maximum of the absolute value of the derivative $T_n'(x)$ can not exceed nL.

For this formulation, to be sure, the credit is not undivided. Bernstein's own statement* asserts merely that $|T_n'(x)|$ can not attain the value $2nL$; his proof is far from simple; and its validity has been called in question,† though this last remark applies only to his discussion of the trigonometric theorem, not to the polynomial case, in which he was primarily interested. The theorem was approached by subsequent writers from different angles, and was finally revealed in its true simplicity by de la Vallée Poussin,‡ whose demonstration is well worth repeating here.

An equivalent form of the assertion to be proved is as follows:

If the maximum of $|T_n'(x)|$ is 1, the maximum of $|T_n(x)|$ can not be less than $1/n$.

Suppose the maximum of $|T_n(x)|$ were less than $1/n$. Then, for any value of the constant C, the function

$$R_n(x) = \frac{1}{n}\sin(nx + C) - T_n(x)$$

would have the sign of $\sin(nx + C)$ at each of the $2n$ points

L'ACADÉMIE ROYALE DE BELGIQUE, (2), vol. 4; the detailed citations below, however, refer to the page-numbers of the essay itself, as printed separately.)

* S. Bernstein 2, p. 20.

† Cf. de la Vallée Poussin, passages cited in next footnote.

‡ de la Vallée Poussin 5, *Sur le maximum du module de la dérivée d'une expression trigonométrique d'ordre et de module bornés*, COMPTES RENDUS, vol. 166 (1918), pp. 843–846; de la Vallée Poussin 1, pp. 39–42.

at which $|\sin(nx + C)|$ has a maximum, in an interval of length 2π. Because of the alternation of these signs, and the periodicity of the functions involved, $R_n(x)$ would then vanish for at least $2n$ distinct values of x in a period, and consequently, by Rolle's theorem, the same thing would be true of the derivative

$$R_n'(x) = \cos(nx + C) - T_n'(x).$$

But if C is chosen so as to make $\cos(nx + C)$ coincide in value with $T_n'(x)$ at a point where the latter is equal to ± 1, $R_n'(x)$ will have a double root at this point. Hence it will have, in an entire period, roots of aggregate multiplicity at least $2n + 1$. For a trigonometric sum of order n, which can not vanish identically—since $R_n(x)$, by reason of its changes of sign, can not be a constant—this is impossible, and the contradiction proves the theorem. It can be shown further,* though not quite so simply, that the maximum of $|T_n(x)|$ will be *greater* than $1/n$, unless $T_n(x)$ has precisely the form $(1/n)\sin(nx + C)$, that is, the latter is the only function for which the limit specified in the theorem is attained.

The corresponding theorem for polynomials may be stated as follows, for a particular interval:

Let $P_n(x)$ be an arbitrary polynomial of degree n, and let L be the maximum of its absolute value for $-1 \leqq x \leqq 1$. Then the maximum of $|\sqrt{1 - x^2}\, P_n'(x)|$ can not exceed nL in the interval specified.

Bernstein established this fact first,† and went from it to the trigonometric case. The relative simplicity of the opposite procedure, on the basis of de la Vallée Poussin's trigonometric proof, has been pointed out by the writer.‡

The most immediate application of the theorem is to a type of argument of which the following is a simple case. Let $f(x)$ be a continuous function of period 2π, and let it be supposed that there exists, for every positive integral value of n, a trigonometric sum $T_n(x)$, of order n, so that

$$|f(x) - T_n(x)| \leqq Q/n^2,$$

where Q is independent of n and x. Then $f(x)$ is the sum of

* Cf. citations in preceding footnote.

† S. Bernstein 2, pp. 6–11.

‡ D. Jackson 2, *On the convergence of certain trigonometric and polynomial approximations*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 22 (1921), pp. 158–166; see p. 162.

the uniformly convergent series

$$T_1 + (T_2 - T_1) + (T_4 - T_2) + (T_8 - T_4) + \cdots,$$

or, if $f(x) - T_n(x) = r_n(x)$,

(1) $f(x) = T_1 + (r_1 - r_2) + (r_2 - r_4) + (r_4 - r_8) + \cdots$.

It will be shown that if the series on the right is differentiated term by term, the resulting series will be uniformly convergent, and $f(x)$ consequently must have a continuous first derivative. Since $|r_n| \leqq Q/n^2$, by hypothesis,

$$|r_1 - r_2| \leqq Q + \frac{Q}{2^2} < 2Q, \quad |r_2 - r_4| \leqq \frac{Q}{2^2} + \frac{Q}{4^2} < \frac{2Q}{2^2},$$

$$|r_4 - r_8| < \frac{2Q}{4^2}, \cdots.$$

But since $r_1 - r_2$ is a trigonometric sum of order 2, etc., it follows from the theorem on the derivative of a trigonometric sum that

$$|(r_1 - r_2)'| < 2 \cdot 2Q, \quad |(r_2 - r_4)'| < 4 \cdot \frac{2Q}{2^2}, \cdots,$$

and, generally, that the derivative of the $(p+1)$th term on the right in (1) does not exceed

$$2^p \cdot \frac{2Q}{2^{2(p-1)}} = \frac{Q}{2^{p-3}}.$$

The last quantity is the general term of a convergent series, and the uniform convergence of the series of derivatives is established. It is evident that this illustration by no means exhausts the possibilities of the method. Thus we may state the following more general theorem.*

If $f(x)$ can be represented by trigonometric sums of order n with an error not exceeding $Q/n^{k+\alpha}$, where k is a positive integer or zero, and $0 < \alpha < 1$, then $f(x)$ has a continuous kth derivative satisfying a Lipschitz condition of order α, that is

$$|f^{(k)}(x') - f^{(k)}(x'')| \leqq \lambda |x' - x''|^{\alpha},$$

where λ is a constant.† The idea can be carried still further.

* Cf. de la Vallée Poussin 1, p. 57; also S. Bernstein 1, and S. Bernstein 2, pp. 22–23, 27.

† The value $\alpha = 1$ is ruled out in the statement of the theorem; but if the hypothesis is satisfied for $\alpha = 1$, as in the preceding illustration ($k = 1$), it is of course satisfied, and the conclusion holds, for an arbitrary value of $\alpha < 1$.

The theorem concerning the order of approximation to $|x|$, like the theorem on the derivative of a trigonometric sum, is most accessible at present through the exposition of de la Vallée Poussin.* The idea of his proof is as follows. In the first place, the problem is referred to the equivalent one of representing $|\sin x|$ by a trigonometric sum of order n in x. Lebesgue† had pointed out that no trigonometric sum of order n can give an error smaller than the corresponding remainder in the Fourier series, multiplied by a quantity of the order of $1/\log n$. As it is readily recognized that the error in the Fourier series for $|\sin x|$ is of the order of $1/n$, the quantity $1/(n \log n)$, already mentioned as a step toward the final result, is seen at once to be an inner limit for the order of the best approximation. If the Fejér mean could be used in the same way as the Fourier sum proper, the desired limit $1/n$ would be obtained with equal ease; for Lebesgue's remark is a consequence of the fact that the partial sum of the Fourier series for an arbitrary function can not exceed the maximum of the absolute value of the function itself, multiplied by a quantity of the order of $\log n$, and in the case of the corresponding Fejér mean, this multiplier is replaced by 1. But the method fails at first, because it is essential also that the Fourier sum of order n for a function $T_n(x)$ is identical with $T_n(x)$, while this is not true of the Fejér mean. However, de la Vallée Poussin observes that if $\tau_k(x)$ is the Fejér mean of order $k-1$, $T_n(x)$ is reproduced identically by the formula

$$T_n(x) = 2\tau_{2n}(x) - \tau_n(x);$$

and from this relation he is able to draw the desired inference, not quite immediately, but by easy steps, in the course of two or three pages. The general theorem which intervenes is as follows.

Let $f(x)$ be a given arbitrary function, let $S_k(x)$ be the partial sum of its Fourier series to terms of order k, and let $T_n(x)$ be any trigonometric sum of order n. Then the maximum of $|f(x) - T_n(x)|$ can not be less than one-fourth the maximum of

$$\left| f(x) - \frac{S_n + S_{n+1} + \cdots + S_{2n-1}}{n} \right|.$$

* de la Vallée Poussin 6, *Sur la meilleure approximation des fonctions d'une variable réelle par des expressions d'ordre donné*, COMPTES RENDUS, vol. 166 (1918), pp. 799–802; de la Vallée Poussin 1, pp. 33–37.

† Cf. e.g. Lebesgue 3, *Sur les intégrales singulières*, ANNALES DE LA FACULTÉ DE TOULOUSE, (3), vol. 1 (1909), pp. 25–117; see pp. 116–117.

Bernstein himself, in the prize essay, proves the theorem about the polynomial representation of $|x|$ in two different ways.* Both of his demonstrations depend on an extension, interesting in itself, of the Tchebychef theory to the problem of the approximate representation of a given continuous function by linear combinations of powers of x with arbitrarily given exponents.† The exponents may be any positive real numbers, not necessarily integral. There is a real generalization, however, even if they are merely selected integers; there would be no point in saying that some powers *may* be omitted, in the approximating polynomials used, for that is always understood, as a matter of course; but the problem is essentially changed if it is demanded that certain powers *shall* be omitted.

By way of obtaining a corresponding generalization of Weierstrass's theorem, Bernstein further inquires under what circumstances a sequence of positive powers $x^{\alpha_1}, x^{\alpha_2}, \cdots$, will be sufficient (in technical language, *complete*) for the uniformly convergent representation of an arbitrary continuous function.‡ He derives a condition which is necessary, and others which are sufficient. Müntz and Szász§ later established a condition which is both necessary and sufficient, namely (except for minor qualifications) the very simple requirement that the series $\Sigma\,(1/\alpha_n)$ diverge. The last-named author extends the discussion to the case of complex exponents.

The rest of Bernstein's memoir must be dismissed with the utmost brevity in the present summary. Among the topics treated may be mentioned some results concerning the absolute value of a polynomial for complex values of the argument,‖ a study of the approximate representation of analytic functions of a complex variable,¶ theorems with regard to the

* S. Bernstein 2, pp. 55–62. See also S. Bernstein 3, *Sur la meilleure approximation de* $|x|$ *par des polynomes de degrés donnés*, ACTA MATHEMATICA, vol. 37 (1914), pp. 1–57.

† S. Bernstein 2, pp. 38–41.

‡ S. Bernstein 2, pp. 78–84.

§ Szász, *Über die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen*, MATHEMATISCHE ANNALEN, vol. 77 (1916), pp. 482–496. The paper of Müntz, which I have not seen, is cited by Szász, loc. cit., p. 483, footnote.

‖ pp. 13–15.

¶ pp. 36, 65–76, 94–95, and elsewhere. This subject, which has an extensive literature of its own, must be left outside the scope of the present review.

degree of convergence of Fourier's series and the Fejér means of Fourier's series,* in the spirit of the following section, and applications of the theory of approximation to questions of the existence of derivatives of functions of more than one variable.†

A paper by Montel‡ presents a noteworthy continuation of Bernstein's work in various directions, particularly with reference to derivatives of fractional order.

6. *Outer Limit of Approximation.* From Picard's inequalities, already cited, for the coefficients in a Fourier series, it follows that, if the function developed has a continuous kth derivative which is of limited variation, the remainder after n terms of the series does not exceed a constant multiple of $1/n^k$. For $k = 1$, the same outer limit of error for the approximation obtainable by means of finite trigonometric sums of order n was found by de la Vallée Poussin,§ under a similar but somewhat more general hypothesis. Most often, however, such discussion has involved the hypothesis of a Lipschitz condition, either on the function itself or on one of its derivatives.

For the case of a function satisfying a Lipschitz condition, de la Vallée Poussin,‖ as has already been noted, found the order of approximation $1/\sqrt{n}$, and the same limit was recorded a little later by Lebesgue,¶ both for polynomials and for trigonometric sums. From the point of view of a rapid survey of the problem, it is not necessary to specify each time which kind of approximating function is used, as the two classes of results are to a large extent interchangeable.

Another step in advance was presently taken by Lebesgue,** who proved that the remainder in the Fourier series for a function of the kind under discussion can not exceed a constant multiple of $(\log n)/n$.

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* pp. 85–93.

† pp. 97–103.

‡Montel, *Sur les polynomes d'approximation*, BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE, vol. 46 (1919), pp. 151–192.

§ de la Vallée Poussin 3.

‖ de la Vallée Poussin 2, p. 222.

¶ Lebesgue 3, pp. 112–115.

** Lebesgue 4, *Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz*, BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE, vol. 38 (1910), pp. 184–210; pp. 199–201.

He also showed that this result is final, as far as the Fourier series is concerned, that is that there exist functions satisfying a Lipschitz condition, for which the remainder does not approach zero any more rapidly.*

There still remained a question as to the degree of approximation that might be attained by a different choice of the approximating functions. The writer showed that a function satisfying a Lipschitz condition can be represented by a trigonometric sum of the nth order,† or by a polynomial of the nth degree,‡ with a maximum error not exceeding a constant multiple of $1/n$, and that the limit $1/n$ in this general statement can not be replaced by any infinitesimal of higher order.§ The truth of the last part of this assertion of course also follows immediately from Bernstein's results with regard to the function $|x|$, which were published a little later. The writer subsequently obtained inequalities for the magnitude of the numerical constants involved, so that the following statements can be made in summary.‖

If $f(x)$ is a function of period 2π satisfying everywhere the condition

$$|f(x') - f(x'')| \leqq \lambda |x' - x''|, \tag{2}$$

it is possible to represent $f(x)$ by a trigonometric sum of order n with a maximum error not exceeding $K\lambda/n$, where K is an absolute constant; the statement is true for $K = 3$, but not for any value of K less than $\pi/2$.

If $f(x)$ satisfies the condition (2) for $a \leqq x \leqq b$, it can be represented throughout this interval by a polynomial of the nth degree with a maximum error not exceeding $L\lambda(b - a)/n$, where L is an absolute constant; the statement is true for $L = 1\frac{1}{2}$, but not for any value of L less than $\frac{1}{2}$.

The upper values for K and L, as actually computed by the writer, were slightly less than 3 and $1\frac{1}{2}$ respectively, and they can be further reduced by other methods, due particularly to

* Lebesgue 4, pp. 202–206.

† D. Jackson 1, Theorem VI; see also top of p. 46.

‡ D. Jackson 1, Theorems I, IV.

§ D. Jackson 1, Theorems XII, XIII.

‖ D. Jackson 3, *On approximation by trigonometric sums and polynomials*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 13 (1912), pp. 491–515; Theorems I, II, VI, VII. There is a misprint in the statement of Theorem VII; for L_2 should be read L_1.

Gronwall.* The ultimate determination of the best values for K and L, however, is still an open question.

The order of approximation being once established for the case of a Lipschitz condition, corresponding results can be deduced without difficulty both for more general and for more restrictive hypotheses. By way of generalization,† let $f(x)$ be an arbitrary continuous function, of period 2π, say, to restrict attention to the trigonometric case, and let $\omega(\delta)$ be the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$. Let $f_1(x)$ be a function represented graphically by a broken line, equal to $f(x)$ at the points $x = 2j\pi/n$ $(j = 0, \pm 1, \pm 2, \cdots)$, and varying linearly from one of these points to the next. Then, for any x, $f(x)$ and $f_1(x)$ will each differ from the value at the nearest vertex point by not more than $\omega(2\pi/n)$, and

$$|f(x) - f_1(x)| \leqq 2\omega(2\pi/n)$$

for all values of x. Since $f_1(x)$ satisfies a Lipschitz condition with coefficient $\omega(2\pi/n)/(2\pi/n)$, there will exist trigonometric sums $T_n(x)$, of order n, so that

$$|f_1(x) - T_n(x)| \leqq 3 \cdot \frac{\omega(2\pi/n)}{2\pi/n} \cdot \frac{1}{n} = \frac{3}{2\pi}\omega(2\pi/n),$$

and

$$|f(x) - T_n(x)| \leqq \left(\frac{3}{2\pi} + 2\right)\omega\left(\frac{2\pi}{n}\right).$$

A similar result can also be obtained directly by means of a definite integral, without the intermediate reference to the Lipschitz condition.‡

On the other hand, suppose that $f(x)$, once more assumed to

* Unpublished letters to the present writer, January 21 and February 16, 1913; *On approximation by trigonometric sums*, this BULLETIN, vol. 21 (1914–15), pp. 9–14. See also de la Vallée Poussin 1, pp. 44–46. My computation, in the Transactions article cited, involved the ratio of two integrals, J_m' and J_m; within a few weeks after its publication, I was in receipt of letters from Messrs. Gronwall, I. Schur, and D. Cauer (the last-named, then a student at Göttingen, writing informally in behalf of Professor Landau), which contained, among other data, two proofs that

$$J_m = \frac{\pi}{3}\left(1 + \frac{1}{2m^2}\right),$$

and seven proofs (including one quoted by Schur from Marcel Riesz) that $\lim_{m=\infty} J_m' = \log 2$.

† See, e.g., D. Jackson 1, Theorem VIII. The fundamental idea of the proof is derived from Lebesgue 4, p. 202.

‡ Cf. de la Vallée Poussin 1, pp. 44–46.

have the period 2π, is not merely continuous itself, but has a continuous first derivative satisfying the Lipschitz condition

$$|f'(x') - f'(x'')| \leqq \lambda |x' - x''|.$$

There is a finite trigonometric sum $T_n'(x)$* such that

$$|f'(x) - T_n'(x)| \leqq \frac{3\lambda}{n}.$$

Moreover, it follows from the proof of the theorem on which this assertion is based, though not from the statement of the theorem itself, that T_n' can be taken so that its constant term is zero,* and its integral therefore is also a trigonometric sum of order n. Let the expression

$$f(0) + \int_0^x T_n'(x)dx$$

be denoted by $T_n(x)$, and let $f'(x) - T_n'(x) = r_n(x)$. Then

$$f(x) = T_n(x) + \int_0^x r_n(x)dx.$$

Of course the integral on the right does not exceed a constant multiple of $1/n$, but it is possible to say much more than that. For the integral has a derivative which does not exceed $3\lambda/n$, that is the integral itself *satisfies a Lipschitz condition with coefficient* $3\lambda/n$, *and can be represented by a trigonometric sum of order* n *with an error not exceeding* $9\lambda/n^2$. This sum, combined with $T_n(x)$, gives an equally close approximation for $f(x)$. In general†—though the proof requires a little attention in detail—*if* $f(x)$ *has a* $(k-1)$*th derivative which satisfies a Lipschitz condition with coefficient* λ, *then* $f(x)$ *can be approximately represented by a trigonometric sum of order* n *with an error not exceeding* $3^k\lambda/n^k$. There is a corresponding but slightly less simple result for polynomial approximation.‡

From the trigonometric theorem it follows immediately, by a remark of Lebesgue already cited, that if $f(x)$ has a $(k-1)$th

* That is, if the function for which an approximate representation is sought, $f'(x)$ in this case, is such that its integral is periodic, the particular approximating function defined in the course of the demonstration will also have a periodic integral.

† D. Jackson 4, *On the approximate representation of an indefinite integral and the degree of convergence of related Fourier's series*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 14 (1913), pp. 343–364; Theorem III. Cf. also D. Jackson 1, Theorem VII, and D. Jackson 3, Theorems III, VIII.

‡ D. Jackson 4, Theorem VII; cf. D. Jackson 1, Theorems II, IVa, and D. Jackson 3, Theorems IV, IX.

derivative satisfying a Lipschitz condition, the remainder after terms of the nth order in the Fourier series for $f(x)$ does not exceed $(\log n)/n^k$, multiplied by a quantity independent of n and x. It is noteworthy that a different method yields a still closer result here, to the extent that the constant multiplier obtained is *independent of* k. If the coefficient in the Lipschitz condition is λ, the error does not exceed $\lambda(\log n)/n^k$, multiplied by an absolute constant.*

Finally, we may note a case in which one of the theorems of the preceding section has an exact converse.† *If* $f(x)$ *has a continuous* k*th derivative satisfying a Lipschitz condition of order* α*, where* k *is a positive integer or zero, and* $0 < \alpha < 1$*, then* $f(x)$ *can be represented by trigonometric sums of order* n *with an error not exceeding* $Q/n^{k+\alpha}$*, where* Q *is independent of* n *and* x. Taken without regard to its converse, the statement can be made both more precise, by exhibiting the dependence of Q on the coefficient in the Lipschitz condition, and more general, by varying the hypothesis on the kth derivative.

7. *Trigonometric Interpolation.* In the theory which forms the subject of § 6, much use is made of formulas involving definite integrals. It is possible to vary the treatment by replacing the integrals by finite sums. From this substitution, which works out most satisfactorily in the trigonometric case, results an extensive theory of trigonometric interpolation.‡ The ordinary formula of interpolation with equidistant ordinates has of course been known for a long time, but its

* Cf. S. Bernstein 2, pp. 92–93; D. Jackson 4, Theorem X; de la Vallée Poussin 1, pp. 23–25, 27–29.

† Cf. de la Vallée Poussin 1, pp. 51–52, 57; also D. Jackson 4, Theorem IV.

‡ Cf., e. g., de la Vallée Poussin 7, *Sur la convergence des formules d'interpolation entre ordonnées équidistantes*, BULLETINS DE L'ACADÉMIE ROYALE DE BELGIQUE, Classe des Sciences, 1908, pp. 319–403; Faber 1, *Über stetige Funktionen*, MATHEMATISCHE ANNALEN, vol. 69 (1910), pp. 372–443, see pp. 417–443; Faber 2, *Über die interpolatorische Darstellung stetiger Funktionen*, JAHRESBERICHT DER DEUTSCHEN MATHEMATIKER-VEREINIGUNG, vol. 23 (1914), pp. 192–210; D. Jackson 5, *On the accuracy of trigonometric interpolation*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 14 (1913), pp. 453–461; D. Jackson 6, *A formula of trigonometric interpolation*, RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO, vol. 37 (1914), pp. 371–375; D. Jackson 7, *Note on trigonometric interpolation*, RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO, vol. 39 (1915), pp. 230–232; D. Jackson 8, *On the order of magnitude of the coefficients in trigonometric interpolation*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 21 (1920), pp. 321–332.

convergence seems to have been first studied in detail by de la Vallée Poussin and Faber within fifteen years. The convergence properties of the interpolating formula are similar to, but not absolutely identical with, those of the Fourier series, which it formally resembles. The analogy extends to questions of degree of convergence, and to the various methods of trigonometric approximation which have been devised for one special purpose or another.

8. *The Method of Least mth Powers.* Of modest importance in itself, perhaps, but of some interest as affording a field for the application of the preceding results, is the theory of what may be called the method of least mth powers.* If $f(x)$ is a continuous function of period 2π (to restrict attention to the trigonometric case once more), it is well known that the trigonometric sum $T_n(x)$, of order n, for which the integral

$$\int_0^{2\pi} [f(x) - T_n(x)]^2 dx$$

has the least possible value, is the partial sum of the Fourier series for $f(x)$. The condition determining $T_n(x)$ can be generalized by writing, in place of the square of the error, the mth power of its absolute value, where m is an arbitrary positive exponent, most conveniently assumed to be greater than 1. The resulting sum $T_n(x)$ is not readily amenable to calculation, because, like the Tchebychef sum, it does not depend linearly on $f(x)$; but it is possible to show that it always exists, is uniquely determined, for $m > 1$, and approaches the Tchebychef sum as a limit when m becomes infinite. Furthermore, if m is held fast and n is allowed to increase indefinitely, $T_n(x)$ will converge uniformly to the value $f(x)$, under suitable hypotheses. The proof depends on Bernstein's theorem concerning the derivative of a trigonometric sum, and on the possibility of representing $f(x)$ by a properly chosen sum with a specified degree of approximation.

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* Cf. D. Jackson 9, *On functions of closest approximation*, TRANSACTIONS OF THE AMERICAN MATH. SOCIETY, vol. 22 (1921), pp. 117–128, and D. Jackson 2, already cited. The polynomial case had previously been treated by Pólya, in a note with which I was not acquainted at the time of writing my own papers: *Sur un algorithme, etc.*, COMPTES RENDUS, vol. 157 (1913), pp. 840–843.

THE EINSTEIN SOLAR FIELD.

BY PROFESSOR LUTHER PFAHLER EISENHART.

(Read before the American Mathematical Society April 23, 1921.)

The Schwarzschild form of the linear element of the Einstein field of gravitation of a mass m at rest with respect to the space-time frame of reference is

$$(1)\quad ds^2=\left(1-\frac{2m}{u_1}\right)dt^2-\frac{u_1}{u_1-2m}du_1^2-u_1^2(du_2^2+\sin^2 u_2\,du_3^2),$$

where t is the coordinate of time, and u_1, u_2, u_3 are space coordinates. Since the coefficients in (1) are independent of t, the particle moves in the 3-space S_3 whose linear element is

$$(2)\quad ds_0^2=\frac{u_1}{u_1-2m}du_1^2+u_1^2(du_2^2+\sin^2 u_2\,du_3^2).$$

If we put

$$(3)\quad \begin{aligned} x_1&=u_1\sin u_2\cos u_3, & x_2&=u_1\sin u_2\sin u_3,\\ x_3&=u_1\cos u_2, & x_4&=4m\sqrt{\frac{u_1}{2m}-1}, \end{aligned}$$

we have

$$(4)\quad ds_0^2=dx_1^2+dx_2^2+dx_3^2+dx_4^2.$$

Hence S_3 is immersed in the euclidean space of four dimensions, S_4, whose rectangular coordinates are x_i $(i=1,\cdots,4)$.* Moreover, as follows from (3), S_3 is the quartic variety defined by

$$(5)\quad x_1^2+x_2^2+x_3^2=\left(\frac{x_4^2}{8m}+2m\right).$$

We shall show that equations (3) define the only three-spread with the linear element (2) in euclidean four-space by making use of the following theorem of Bianchi:† In euclidean n-space $(n>3)$ every hypersurface is not deformable unless at least $n-2$ of the principal radii of curvature are infinite.

* Cf. Kasner, American Journal of Mathematics, vol. 43, p. 132 (April, 1921).

† *Lezioni*, vol. 1, p. 467.

In fact we show that all of the principal radii of curvature of (5) are finite. If X_i denote the direction-cosines of the normal to (5), that is

$$\sum_{i=1}^{4} X_i \frac{\partial x_i}{\partial u_j} = 0 \ (j = 1, 2, 3), \qquad \sum X_i^2 = 1,$$

then

$$X_1 = \sqrt{\frac{2m}{u_1}} \sin u_2 \cos u_3, \qquad X_2 = \sqrt{\frac{2m}{u_1}} \sin u_2 \sin u_3,$$

$$X_3 = \sqrt{\frac{2m}{u_1}} \cos u_2, \qquad X_4 = -\sqrt{1 - \frac{2m}{u_1}}.$$

If we define functions Ω_{rs} by

$$\Omega_{rs} = -\sum_i \frac{\partial X_i}{\partial u_r} \frac{\partial x_i}{\partial u_s},$$

we find $\Omega_{rs} = 0 \ (r \neq s)$ and

$$\Omega_{11} = \frac{1}{u_1 - 2m} \sqrt{\frac{m}{2u_1}}, \qquad \Omega_{22} = -\sqrt{2mu_1}, \qquad \Omega_{33} = -\sqrt{2mu_1} \sin^2 u_2.$$

The principal radii of curvature are given by*

$$\frac{1}{R_1} = \frac{\Omega_{11}}{\dfrac{u_1}{u_1 - 2m}} = \sqrt{\frac{m}{2u_1^3}}, \qquad \frac{1}{R_2} = \frac{\Omega_{22}}{u_1^2} = -\sqrt{\frac{2m}{u_1^3}},$$

$$\frac{1}{R_3} = \frac{\Omega_{33}}{u_1^2 \sin^2 u_2} = \sqrt{\frac{2m}{u_1^3}},$$

which are finite since $u_1 \neq 0$.

In accordance with the Einstein theory the world-line of a particle in the gravitational field is a geodesic of the space with the linear element (1), that is a curve along which $\int ds$ is stationary; and the world-line of a ray of light is a curve for which $ds = 0$ and $\int dt$ is stationary. In each case the frame of reference can be so chosen that a particular path satisfies the condition $u_2 = \pi/2$.† From (3) it follows that for this path $x_3 = 0$, and hence the path considered by astronomers is the projection upon the plane $x_3 = 0$ of a curve on the surface

$$x_1^2 + x_2^2 = \left(\frac{x_4^2}{8m} + 2m\right)^2.$$

* Bianchi, l.c., pp. 368, 472.

† Cf. Eddington, *Report on the Relativity Theory of Gravitation*, p. 49.

This is the surface of revolution of a parabola of latus rectum $8m$ about its directrix. A similar result was obtained by Flamm* who considered the surface, in euclidean three-space, for which the linear element is given by (2) for $u_2 = \pi/2$.

PRINCETON UNIVERSITY,
April 16, 1921.

A COVARIANT OF THREE CIRCLES.

BY PROFESSOR A. B. COBLE.

(Read before the American Mathematical Society April 23, 1921.)

Dr. J. L. Walsh † has stated the following theorem.

THEOREM. *If the double ratio,* $(z_1, z_3 \mid z_2, z)$, *of the four points* z_1, z_2, z_3, z *in the complex plane is a real number* λ, *then as the points* z_1, z_2, z_3 *run over the circles* C_1, C_2, C_3 (*and their interiors*) *respectively, the locus of* z *is a circle* (*and its interior*).

This locus is evidently a covariant, under the inversive group, of the three given circles, which is rational in λ. We find in (8) its equation and incidentally prove the theorem.

In conjugate coordinates $z, \bar{z}$, a circle is

$$C_1(z) = a_1 z\bar{z} + \alpha_1 z + \bar{\alpha}_1 \bar{z} + b_1 = 0,$$

where a_1, b_1 are real, and α_1, $\bar{\alpha}_1$ are conjugate imaginary. The bilinear invariant of two circles $C_1(z)$, $C_2(z)$ is

$$[C_1, C_2] = \alpha_1\bar{\alpha}_2 + \alpha_2\bar{\alpha}_1 - a_1 b_2 - a_2 b_1.$$

It vanishes when the two circles are orthogonal. When they coincide it becomes $[C_1 C_1] = 2(\alpha_1\bar{\alpha}_1 - a_1 b_1)$. This vanishes when C_1 is a *point circle,* i.e. one whose equation is

$$P_{z_i}(z) = (z - z_i)(\bar{z} - \bar{z}_i) = 0. \tag{1}$$

It is easily verified that

$$[C_1, P_{z_i}(z)] = -C_1(z_i); \quad [P_{z_i}(z), P_{z_k}(z)] = -P_{z_i}(z_k) = -P_{z_k}(z_i).$$

The two point circles of the pencil $C(z) + \mu K(z) = 0$ are determined by

$$[C + \mu K, C + \mu K] = [C, C] + 2\mu\,[CK] + \mu^2\,[KK] = 0.$$

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* PHYSIK. ZEITSCHR., vol. 17 (1916), p. 449.

† TRANSACTIONS AMER. MATH. SOCIETY, vol. 22 (1921), p. 101. The geometric proof of this theorem given by Dr. Walsh is very complicated. The method of proof followed here is considered by Dr. Walsh (loc. cit.,

They coincide and the circles $C(z)$, $K(z)$ touch when

$$[KC]^2 - [KK]\,[CC] = 0.$$

We begin the proof with the condition

$$\frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)} = \lambda, \tag{2}$$

and set

$$l_1 = (z_2 - z_3)(z - z_1),\quad l_2 = (z_3 - z_1)(z - z_2),\quad l_3 = (z_1 - z_2)(z - z_3),$$

$$l_1 + l_2 + l_3 = 0; \tag{3}$$

$$q_1 = \lambda(\lambda - 1),\qquad q_2 = \lambda,\qquad q_3 = 1 - \lambda,$$

$$q_2q_3 + q_3q_1 + q_1q_2 = 0. \tag{4}$$

The condition (2), or $(\lambda + l_3/l_1) = 0$, when multiplied by its conjugate, $(\lambda + \bar{l}_3/\bar{l}_1)$, is easily reduced by the use of (3) and (4) to the symmetrical form $q_1l_1\bar{l}_1 + q_2l_2\bar{l}_2 + q_3l_3\bar{l}_3 = 0$. Again, in the notation of (1), this condition is

$$q_1P_{z_2}(z_3)\cdot P_{z_3}(z) + q_2P_{z_3}(z_1)\cdot P_{z_1}(z) + q_3P_{z_1}(z_2)\cdot P_{z_2}(z) = 0.$$

For fixed values of λ, z_2, z_3, z this is the equation which determines z_1. If z_1 lies on a circle $C_1(z) = 0$, then

$$K \equiv q_1P_{z_2}(z_3)\cdot C_1(z) + q_2C_1(z_3)\cdot P_{z_3}(z) + q_3C_1(z_2)\cdot P_{z_2}(z) = 0.$$

For fixed z_2, z_3, and z_1 variable in the circle C_1, $K(z) = 0$ is the equation of the circle within which z lies. Now let z_2 range from its fixed position outward in all directions toward the boundary of a circle C_2. Then the circle $K(z)$ ranges outward in all directions from its original position toward the boundary of an envelope which is the *outer part* of the envelope of the ring of circles $K(z)$ as z_2 runs over the circumference of the circle C_2. This envelope is the locus of points z for which K *regarded as a circle in the variables* z_2, $\bar{z}_2$ touches the given circle C_2 and therefore the equation of the envelope is

$$[KC_2]^2 - [KK][C_2C_2] = 0.$$

We shall show that $[KK]$ is a perfect square and therefore the envelope factors into a pair of circles of which we want the outer. We notice that $K(z_2)$ breaks up into three terms, $K_1(z_2) + K_2(z_2) + K_3(z_2)$. Hence

$$[KK] = \Sigma[K_iK_i] + 2\Sigma[K_iK_j] \quad (i, j = 1, 2, 3;\ i \neq j).$$

footnote, p. 102) but rejected because of algebraic difficulties. These however are not inherent. The algebraic method has, moreover, the decided advantage of furnishing the required envelope in covariant form.

To within the coefficients q the terms $[K_iK_j]$ are all alike and equal to $-P_{z_3}(z)\cdot C_1(z)\cdot C_1(z_3)$. But according to (4) the sum of the coefficients of these three terms vanishes. Moreover $[K_iK_i] = 0$ $(i = 1, 2)$ since $K_1(z_2)$ and $K_2(z_2)$ are point circles. Hence $[KK] = q_3^2\cdot(P_z{}^3(z))^2\cdot[C_1C_1]$. Also

$$[KC_2] = -q_1C_1(z)\cdot C_2(z_3) - q_2C_1(z_3)\cdot C_2(z) + q_3P_{z_3}(z)\cdot[C_1C_2].$$

Thus for proper choice of the sign of the radicals the outer part of the envelope is the circle

$$(5)\quad L \equiv -q_1C_1(z)\cdot C_2(z_3) - q_2C_1(z_3)\cdot C_2(z) + q_3P_{z_3}(z)\{[C_1C_2] - \sqrt{[C_1C_1]}\ \sqrt{[C_2C_2]}\}.$$

We now let z_3 run over a circle $C_3(z) = 0$. As before the envelope of the circle $L(z)$ in (5) is the tact-invariant of L regarded as a circle in the variable z_3 and of $C_3(z_3) = 0$. It is therefore

$$(6)\qquad [LC_3]^2 - [LL]\,[C_3C_3] = 0.$$

Again the term $[LL]$ is a perfect square. In fact

$$[LL] = q_1{}^2C_1{}^2(z)\cdot[C_2C_2] + q_2{}^2C_2{}^2(z)\cdot[C_1C_1] + 2q_1q_2C_1(z)\cdot C_2(z)\cdot[C_1C_2] + 2q_3(q_1+q_2)C_1(z)\cdot C_2(z)\{[C_1C_2] - \sqrt{[C_1C_1]}\ \sqrt{[C_2C_2]}\},$$

the term in $q_3{}^2$ dropping out since q_3 is the coefficient of a point circle. Since $q_3(q_1+q_2) = -q_1q_2$ this becomes

$$(7)\qquad [LL] = \{q_1C_1(z)\sqrt{[C_2C_2]} + q_2C_2(z)\sqrt{[C_1C_1]}\}^2.$$

Hence the final envelope (6) factors into two circles (necessarily inner and outer) and the theorem is proved.

In order to obtain the equation of the envelope we note that

$$[LC_3] = -q_1C_1(z)\cdot[C_2C_3] - q_2C_2(z)\cdot[C_1C_3] - q_3C_3(z)\cdot\{[C_1C_2] - \sqrt{[C_1C_1]}\ \sqrt{[C_2C_2]}\}.$$

This, together with (7), yields the factors of (6), whence

The locus of z referred to in the theorem is, in explicit form,

$$(8)\quad \begin{aligned}&\lambda(\lambda-1)\cdot C_1(z)\{[C_2C_3] - \sqrt{[C_2C_2]}\ \sqrt{[C_3C_3]}\}\\ &+\lambda\cdot C_2(z)\{[C_3C_1] - \sqrt{[C_3C_3]}\ \sqrt{[C_1C_1]}\}\\ &+(1-\lambda)\cdot C_3(z)\{[C_1C_2] - \sqrt{[C_1C_1]}\ \sqrt{[C_2C_2]}\} = 0;\end{aligned}$$

where the sign of the radical $\sqrt{[C_iC_i]}$ is to be taken opposite the sign of the quadratic q_i for given λ.

On account of the symmetry and homogeneity of this result the verification of sign can be made for (5) and $a_1 = a_2 = 1$. We have in (5) two circles which determine the pencil

$$(9)\quad \mu\{-q_1C_2(z_3)\cdot C_1(z)-q_2C_1(z_3)\cdot C_2(z)+q_3P_{z_3}(z)\cdot[C_1C_2]\}$$
$$-\{q_3\sqrt{[C_1C_1]}\sqrt{[C_2C_2]}P_{z_3}(z)\}=0.$$

Let C_1, C_2 be small circles around the points z_1, z_2 respectively which themselves are not on a line with the point z_3. The pencil (9) contains the point circle $P_{z_3}(z)$ and therefore another point circle P interior to the two point circles (5) since $P_{z_3}(z)$ is exterior to them. Hence in (9) we must have for $\mu=0$ the point circle z_3; for $\mu=\mu_R$ the radical axis; for $\mu=1$, the outer circle (5); for $\mu=-1$ the inner circle (5); and for $\mu=c(-1<c<0)$, the point circle P. Thus μ_R, the parameter of the radical axis of the pencil (9) must be positive. But $\mu_R=q_3\sqrt{[C_1C_1]}\sqrt{[C_2C_2]}/(-q_1C_2(z_3)-q_2C_1(z_3)+q_3[C_1C_2])$. If now the circles C_1, C_2 approach the points z_1, z_2 as limits the denominator of μ_R approaches as a limit

$$(10)\qquad -(q_1\alpha^2+q_2\beta^2+q_3\gamma^2)$$

where α, β, γ are the lengths of the sides opposite the vertices z_1, z_2, z_3 of the triangle z_1, z_2, z_3. In terms of λ (10) becomes

$$(11)\qquad -\alpha^2\lambda^2+(\gamma^2+\alpha^2-\beta^2)\lambda-\gamma^2.$$

The discriminant of (11) is

$$(\alpha+\beta+\gamma)(-\alpha+\beta+\gamma)(-\beta+\alpha+\gamma)(\gamma-\alpha-\beta)$$

which is negative. Hence (11) is a definite quadratic form evidently negative for sufficiently large λ. Then (10) is negative for all real values of λ and this requires that $q_3\sqrt{[C_1C_1]}\sqrt{[C_2C_2]}$ be negative. Since $q_1q_2q_3=-\lambda^2(\lambda-1)^2$ is negative for all real values of λ, the three radicals must take the same signs as, or opposite signs to, the three quadratics q,

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ON SKEW PARABOLAS.

BY DR. MARY F. CURTIS.

The theorem that a real rectifiable skew parabola is a helix, proved in my note in this BULLETIN, November, 1918, for skew parabolas which can be represented in rectangular coordinates by equations of the form:

$$(1)\qquad x_1=at,\qquad x_2=bt^2,\qquad x_3=ct^3,\qquad abc\neq 0,$$

was extended by Professor Hayashi in this BULLETIN, November, 1919, to cover all real skew parabolas, whose equations he reduces to the form

(2) $x_1 = a_1t^3 + a_3t,\ x_2 = b_1t^3 + b_2t^2,\ x_3 = c_1t^3,\ a_3b_2c_1 \neq 0.$

Professor Loria, in an extract from a letter to Professor D. E. Smith published in this BULLETIN, February, 1921, states that Professor Hayashi's work was unnecessary, in that every (real) skew parabola can be represented in rectangular coordinates by equations of the form (1).

When the coordinates are oblique, (1) does represent every real skew parabola.* Professor Loria's statement that this is equally true when the coordinates are rectangular is, however, at fault. A skew parabola C:

(3) $$x_i = a_it^3 + b_it^2 + c_it + d_i \qquad (i = 1, 2, 3)$$

lies on a unique parabolic cylinder S and meets the ruling R on which the vertices of the orthogonal sections of S lie in a unique finite point P. If $P : t = 0$ is taken as the origin of coordinates, R as the x_3-axis, and the axis of the orthogonal section through P as the x_2-axis, equations (3) become

(4) $x_1 = c_1t,\ x_2 = b_2t^2,\ x_3 = a_3t^3 + b_3t^2 + c_3t,\ a_3b_2c_1 \neq 0,$

and these are as simple equations representing a general skew parabola in rectangular coordinates as can be found.†

It is clear that the osculating plane at $P : t = 0$ is the plane $x_3 = 0$ if and only if $b_3 = c_3 = 0$. Then (4) reduces to (1).

THEOREM. *A real skew parabola C can be represented in rectangular coordinates by equations of the form* (1) *if and only if, at the point in which C meets the line of vertices of the parabolic cylinder on which C lies, the osculating plane of C is orthogonal to the cylinder.*

The representation (4) for the general skew parabola, the theorem and the representation (1) for the particular skew parabolas which the theorem singles out hold in the complex domain, unless the rulings of the cylinder S are minimal, or the parallel planes each of which cuts S in a single ruling are minimal, or both the rulings and the planes are minimal. In such cases simple representations of the skew parabola exist.

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* ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN, III, C 2: O. Staude. *Flächen* 2. *Ordnung und ihre Systeme*, p. 234. ENCYCLOPÉDIE DES SCIENCES MATHÉMATIQUES III, 22 (1914), p. 130.

† O. Staude: *Analytische Geometrie der Kubischen Kegelschnitte* (1913), p. 139. The equations (2) which refer the curve to the tangent, principal normal and binormal at an arbitrary point may, for some purposes, be preferable to the equations (4).

THE DISPERSION OF OBSERVATIONS.

BY PROFESSOR JULIAN LOWELL COOLIDGE.

In the study of statistics bearing upon groups of objects, it is of fundamental importance to know whether, and to what extent, the different objects in the same group are comparable, and also to what extent one group is comparable with another. The first writer to study such questions seems to have been Lexis* and subsequent writers have developed his ideas of normal, supernormal and subnormal dispersion, or of Bernoulli, Lexis, and Poisson series.

In all articles dealing with these topics, which have come to the writer's attention, the data are either restricted to series of probabilities or frequency ratios, or else it is assumed that variations are distributed according to the Gaussian law of error. This restriction is unnecessary and unfortunate. The purpose of the present paper is to show how any series of sets of observations may be tested for the normality of their dispersion. The mathematical means employed are of the most elementary nature.

Suppose that we have n observed values $y_1, y_2, \cdots, y_n$. The expression

$$\sqrt{\frac{1}{n}\Big[\sum_i \Big(y_i - \sum_j \frac{y_j}{n}\Big)^2\Big]}$$

is called the *dispersion* or *standard deviation* of the set, and is of first importance in the theory of errors of observation, and of statistics. Let us find the mean value of its square. For simplicity, the mean value of y_i shall be called a_i while the mean value of its square is A_i. We shall write the averages,

$$(1/n)\sum y_i = y, \qquad (1/n)\sum a_i = a.$$

The square of the dispersion is

$$(1/n)\sum_i (y_i - y)^2 = (1/n)\sum_i \Big\{\Big((y_i - a_i) - (y - a)\Big) + [a_i - a]\Big\}^2.$$

In the large round parenthesis, if the averages y and a be replaced by their expanded values, we have the sum of a number of terms, each of which has the mean value 0, and as these terms are independent, the mean value of the product

* *Zur Theorie der Massenerscheinungen in der menschlichen Gesellschaft*, Freiburg, 1877.

of two is also 0. The mean value of the square of the round bracket is the sum of the mean values of the squares of its individual terms. The mean value of the product of the round and square brackets is 0, and the mean value of the square of the square bracket is its ostensible value.

$$(y_i - a_i) - (y - a) = y_i - a_i - \frac{\sum(y_i - a_i)}{n}$$

$$= \frac{(n-1)}{n}(y_i - a_i) - \frac{(y_j - a_j) + (y_k - a_k) + \cdots (y_n - a_n)}{n}.$$

The mean value of $(y_i - a_i)^2$ is $A_i - a_i^2$, and the coefficient will be $(n - 1/n)^2$ in one case, and $1/n^2$ in $(n-1)$ other cases. Summing, we reach the following theorem.

FUNDAMENTAL DISPERSION THEOREM. *If n independent quantities $y_1, y_2, \cdots, y_n$ be given, their mean values being $a_1, a_2, \cdots, a_n$, while the mean values of the squares are $A_1, A_2, \cdots, A_n$ respectively, and if $y = (1/n)\Sigma_i y_i$, $a = (1/n)\Sigma_i a_i$, then the mean value of the square of the dispersion is*

$$\frac{1}{n}\left[\frac{n-1}{n}\sum_i(A_i - a_i^2) + \sum_i(a_i - a)^2\right].$$

In practice it is convenient, first to replace $(n-1)/n$ by 1, and, secondly, to replace the mean value of the square of the dispersion by its observed value. We thus obtain the approximate equation

$$(1) \qquad \sum_i(y_i - y)^2 = \sum_i(A_i - a_i^2) + \sum_i(a_i - a)^2.$$

Let us show the practical application of this equation. Suppose that we have N sets, each of s observations,

$$\begin{aligned} &x_{11}\ x_{12}\ \cdots\ x_{1s}; && \sum_j x_{1j} = x_1, \\ &x_{21}\ x_{22}\ \cdots\ x_{2s}; && \sum_j x_{2j} = x_2, \\ &\cdots\ \cdots\ \cdots\ \

Again, we may write $(x_i - a_i) = \Sigma_j(x_{ij} - a_{ij})$. The mean value of $(x_i - a_i)^2$ is $\Sigma_j(A_{ij} - a_{ij}^2)$. Hence applying (1) once more, we find

$$(3) \qquad \sum_i (x_i - x)^2 = \sum_{i,j} (A_{ij} - a_{ij}^2) + \sum_i (a_i - a)^2.$$

Eliminating $\Sigma_{ij}(A_{ij} - a_{ij}^2)$ between (2) and (3), we obtain

$$(4) \quad \sum_{i,j} \left[\left(x_{ij} - \frac{x_i}{s} \right)^2 - \left(a_{ij} - \frac{a_i}{s} \right)^2 \right] = \sum_i [(x_i - x)^2 - (a_i - a)^2] .$$

This is our fundamental equation. In practice it is usual to see whether a given series of sets of observations belong to one of three recognized types:

(A) BERNOULLI SERIES. Here all of the observations are supposed to be upon the same object, or at least the mean value does not vary from one object to another. We have

$$a_{ij} \equiv a_i, \quad a_i \equiv a, \quad \sum_{i,j} \left(x_{ij} - \frac{x_i}{s} \right)^2 = \sum_i (x_i - x)^2.$$

Such a series is said to have *normal dispersion.*

(B) LEXIS SERIES. All observations in one set are supposed to bear on the same object, but the object varies from set to set.

$$a_{ij} \equiv a_i, \quad a_i \not\equiv a, \quad \sum_{i,j} \left(x_{ij} - \frac{x_i}{s} \right)^2 < \sum_i (x_i - x)^2.$$

Such a series is said to have *supernormal dispersion.*

(C) POISSON SERIES. The objects within a set are supposed to differ from one another, but all sets are supposed comparable.

$$a_{ij} \not\equiv a_i, \quad a_i \equiv a, \quad \sum_{i,j} \left(x_{ij} - \frac{x_i}{s} \right)^2 > \sum_i (x_i - x)^2.$$

Such a series is said to have *sub-normal dispersion.*

What we can do in practice is this. *We calculate the quantities* $\Sigma_{i,j}\,(x_{ij} - x_i/s)^2$ *and* $\Sigma_i(x_i - x)^2$. *If they be virtually equal, we are sure that the members of each set can not all be equal, unless the sets are all the same, and vice versa. If the first be less than the second, the various sets can not be the same. If the first be greater than the second, the individuals must differ within a set.*

It is well, in conclusion, to show how our formulas connect with the usual formulas for the dispersion of ratios. The problem is to see whether certain frequency ratios vary from case to case, or from set to set. If the generic letter for one of our probabilities be p_{ij}, and this represent the chance that

x_{ij} takes the value 1, while in the contrary case it takes the value 0, then a_i/s represents the average probability for the ith set.

$$\sum_j p_{ij} = sp_i = a_i \qquad \sum_i p_i = Np = a/s \qquad p + q = 1.$$

$A_{ij} - a_{ij}^2 =$ Mean value of $(x_{ij} - p_{ij})^2 = p_{ij} - p_{ij}^2$.

From (3), we find

$$\sum_i (x_i - x)^2 = \sum_{i,j} (p_{ij} - p_{ij}^2) + s^2 \sum_i (p_i - p)^2.$$

Now

$$\sum_j (p_{ij} - p_i)^2 = \sum_j p_{ij}^2 - sp_i^2.$$

Hence

$$\sum_j p_{ij}^2 = sp_i^2 + \sum_j (p_{ij} - p_i)^2.$$

Similarly

$$\sum_i p_i^2 = Np^2 + \sum_i (p_i - p)^2,$$

$$\sum_i (x_i - x)^2 = \sum_i [sp_i - sp_i^2 - \sum_j (p_{ij} - p_i)^2 + s^2(p_i - p)^2]$$

$$= Nspq - \sum_{i,j} (p_{ij} - p_i)^2 + (s^2 - s) \sum_i (p_i - p)^2,$$

$$(5) \quad \frac{1}{N} \sum_i (x_i - x)^2 = spq - \frac{1}{N} \sum_{i,j} (p_{ij} - p_i)^2 + \frac{s^2 - s}{N} \sum_i (p_i - p)^2.$$

To take up the particular cases:

(A) BERNOULLI SERIES. The probability is constant throughout.

$$p_{ij} \equiv p_i \equiv p, \quad \frac{1}{N} \sum_i (x_i - x)^2 = spq.$$

(B) LEXIS SERIES. The probability is constant throughout a set, but varies from set to set.

$$p_{ij} \equiv p_i, \quad p_i \not\equiv p. \quad \frac{1}{N} \sum_i (x_i - x)^2 = spq + \frac{s^2 - s}{N} \sum_i (p_i - p)^2.$$

(C) POISSON SERIES. The probability varies within the set, but all sets are comparable.

$$p_{ij} \not\equiv p_i, \quad p_i \equiv p. \quad \frac{1}{N} \sum_i (x_i - x)^2 = spq - \sum_j (p_{ij} - p_i)^2.$$

These are the standard formulas.*

HARVARD UNIVERSITY,
April, 1921.

* Conf. Fisher, *Mathematical Theory of Probabilities*, New York, 1915, pp. 120 ff.

THE ISOMORPHISMS OF COMPLEX ALGEBRA.

BY DR. NORBERT WIENER.

(Read before the American Mathematical Society December 28, 1917.)

Definitions and Conventions. By the *transform* by a function $\phi(x)$ of a function $F(x_1, x_2, \cdots, x_n)$ we mean the expression

$$\varphi\{F(\varphi^{-1}(x_1), \varphi^{-1}(x_2), \cdots, \varphi^{-1}(x_n))\}.$$

We shall consider ∞ as a possible argument for a function, and we shall define $F(\infty)$ to be $\lim_{x=\infty} F(x)$. Similarly if $\lim_{x=\infty} F(x) = \infty$, as $|x|$ grows without limit in every possible way, we shall say that $F(\infty)$ is ∞, while if $\lim_{x=a} F(x) = \infty$ as x approaches a in every possible way, we shall say that $F(a) = \infty$. Similar conventions will be established for functions of more than one variable.

A function F of the complex variable z is said to be *continuous* at z_1 if $\lim_{|\epsilon|=0} F(z_1 + \epsilon) = F(z_1)$.

A function $F(x_1, x_2, \cdots, x_n)$ will be said to be *continuous in general* if there are only a finite number of sets $(x_1, x_2, \cdots, x_n)$ for which $F(x_1, x_2, \cdots, x_n)$ fails to be continuous.

A variable μ is said to *depend uniquely* on $x_1, x_2, \cdots, x_n$ if there are only a finite number of sets $x_1, x_2, \cdots, x_n$ for which μ is not uniquely determined, and if for these μ is undefined.

THEOREM. *If a function Φ is single-valued, as well as its inverse, over the set of arguments consisting of all complex numbers and ∞, and if it is continuous in general, and transforms every algebraic function into an algebraic function, it is a linear function or its conjugate.* The proof of this will involve almost no considerations except those of elementary algebra.

To begin with, let us consider any function of the form

$$F(x, y) = \frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}$$

which does not degenerate into a constant nor into a function of a single variable. Such a function has the following properties.

(1) $F(x, y)$ depends uniquely on x and y.
(2) x depends uniquely on y and $F(x, y)$.
(3) y depends uniquely on x and $F(x, y)$.

It is easy to show that these properties will also belong to any transform of F by a biunivocal function

Now, no algebraic function not of the form

$$\frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}$$

has properties (1), (2), and (3). Any algebraic function with these properties must be obtained by solving for z an equation $P(x, y, z) = 0$, where P is some polynomial. Since z is uniquely determined by x and y, we may assume, without any real loss of generality, that P is of the form*

$$[g(x, y) + zh(x, y)]^m,$$

which we may write

$$g^m + mg^{m-1}hz + \text{terms in higher powers of } z.$$

Since P is a polynomial, g^m and $g^{m-1}h$ are polynomials. Let us call these Q and R, respectively. P is then of the form

$$\frac{1}{g^{m-m}}\{g^m + zhg^{m-1}\}^m = \psi(x, y)\{Q(x, y) + zR(x, y)\}^m,$$

where Q and R may be taken so as to be mutually prime, by removing any common factor and transferring it to the factor ψ. By considerations of symmetry, we may write

$$\begin{aligned} P(x, y, z) &= \psi'(y, z)\{Q'(y, z) + xR'(y, z)\}^{m'} \\ &= \psi''(x, z)\{Q''(x, z) + yR''(x, z)\}^{m''}. \end{aligned}$$

It follows from a consideration of the irreducible factors of P that we may write

$$P(x, y, z) = (\alpha + \beta x + \gamma y + \delta xy - \epsilon z - \xi xz - \eta yz - \vartheta xyz)^m,$$

where $\alpha, \cdots, \vartheta$ are constants. Hence the only algebraic functions satisfying conditions (1), (2), and (3) are of the form

$$F(x, y) = \frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}.$$

We shall now state a lemma which may readily be established by the usual method of undetermined coefficients. The function $1 - x/y$ is the only function of the form

$$F(x, y) = \frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}$$

which satisfies the four conditions

* Notice that this is only true in complex algebra.

(1) $F(x, x) = 0$ for all x other than 0 or ∞;
(2) $F(0, x) = 1$ for all x other than 0 or ∞;
(3) $F(x, 0) = \infty$ for all x other than 0 or ∞;
(4) $F\{1, F\{1, F(1, x)\}\} = x$ for all x other than 0 or ∞.

Now, consider a function $G(x, y)$ which is derived from $1 - x/y$ by a transformation Φ which is continuous in general, which is one-to-one, and which leaves all algebraic functions algebraic. We have already shown that any such function $G(x, y)$ must be of the form

$$\frac{\alpha + \beta x + \gamma y + \delta xy}{\epsilon + \xi x + \eta y + \vartheta xy}.$$

Let us now subject this function to a linear transformation φ which turns the transforms by Φ of 0, 1, and ∞ back into these respective numbers. The resulting function, which we shall call $H(x, y)$, satisfies conditions (1), (2), (3), and (4). Hence we have $H(x, y) = 1 - x/y$. Hence the transformation χ formed by performing first Φ and then φ leaves invariant the function $1 - x/y$. I have shown in an earlier paper* that addition and multiplication can be obtained by the iteration of the function $1 - x/y$. Hence the transformation χ leaves these functions invariant.

Now, any continuous transformation of the complex plane which leaves multiplication invariant must leave invariant the circle of the roots of unity. In a like manner, any continuous transformation of the number-plane that keeps addition invariant must keep invariant first the set of all sets each consisting of all the rational multiples of some number, then the set of all lines each consisting of the real multiples of some number, and finally must turn into a line every line in the complex plane, since every such line can be formed by adding the same number to the product of a given complex number by a variable real number. Hence our transformation χ is an affine transformation which keeps invariant the points 0, 1, and the unit circle. There is no difficulty in showing that any such transformation is either the identity or the conjugate transformation.

It follows that Φ is obtainable by applying after the identity or the conjugate operation the linear operation φ^{-1}. This is precisely the theorem which we set out to prove.

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* *Bilinear operations generating all operations rational in a domain* Ω, ANNALS OF MATHEMATICS, March, 1920.

ON THE GENERALIZATION OF CERTAIN FUNDAMENTAL FORMULAS OF THE MATHEMATICAL THEORY OF FINANCE.

BY PROFESSOR C. H. FORSYTH.

(Read before the American Mathematical Society December 29, 1920.)

1. *Introduction.* Certain of the fundamental formulas in the mathematical theory of finance lack a generality which would add much to the usefulness of such formulas. The lack of generality is due to an assumption that the periodic returns from capital profitably invested will be invested at the same rate of interest at which the principal itself is invested. For example, the most fundamental formula in the mathematical theory of finance,

$$S = P(1 + i)^n,$$

is employed to give the amount to which a principal P will accumulate at the end of n years at compound interest at rate i. But it is assumed in its derivation that every payment of interest upon the principal will be immediately invested, that all the interest upon this interest will be immediately invested, that all the interest on this third system of interest will be immediately invested, and so on, all at the original rate of interest. A much more general formula would be obtained both in this case and in many other cases if an allowance were made for the fact that the interest upon the principal may be invested at another rate of interest.

The purpose of this paper is to derive various fundamental formulas based upon an assumption that the periodic returns from an investment will be invested at another rate of interest. A few formulas will also be given which assume that even the interest upon the investment of the regular interest payments, and all further interest income, will be invested at a third rate of interest, such as that of a savings bank. Such formulas will, of course, include as special cases the formulas ordinarily employed and even the cases where the interest payments are not invested at all.

2. *The Amount of an Investment of a Single Sum. Two Rates of Interest.* If P dollars are placed at interest at rate i for n years the annual payments of interest will constitute an

annuity of iP per year. If we assume that these payments and all further interest will be immediately invested at a rate r the total accumulations or *amount* at the end of n years will evidently be

$$S = P(1 + is_{\overline{n}|}), \tag{1}$$

to be valued at rate r. If it should prove possible to invest all interest payments at the rate of interest at which the original principal was invested, that is if $r = i$, the formula (1) reduces at once to the usual form $S = P(1 + i)^n$.

If the regular interest payments are not invested at all, that is if $r = 0$, formula (1) reduces to the form $S = P(1 + ni)$, which is the formula employed in simple interest.

If the regular interest payments at rate i were made m times a year and the interest at rate r upon these interest payments were paid and compounded t times a year the formula for the amount at the end of n years would be

$$S = P(1 + js_{\overline{n}|}^{(m)}), \tag{2}$$

to be valued at rate r/t, where j is the nominal rate of interest realized upon the original principal, and $s_{\overline{n}|}^{(m)}$ is the symbol generally adopted for the conventional amount of an annuity of 1 per annum but payable m times a year, that is,

$$s_{\overline{n}|}^{(m)} = \frac{1}{m} \frac{\left(1 + \frac{r}{t}\right)^{tn} - 1}{\left(1 + \frac{r}{t}\right)^{t/m} - 1},$$

at rate r/t, or

$$s_{\overline{n}|}^{(m)} = \frac{1}{m} \frac{s_{\overline{tn}|}}{s_{\overline{t/m}|}}, \tag{A}$$

at any rate r/t. In case t/m should prove to be fractional, the value of $1/s_{\overline{t/m}|}$ could be found in a table of values of $1/s_{\overline{n}|}$ for fractional values of n or in a table of values of $i/j_{(p)}$, where

$$\frac{1}{s_{\overline{1/p}|}} = p \cdot \frac{i}{j_{(p)}}.$$

If it were found possible to invest all the interest payments at the same rate of interest and at the same frequency of conversion at which the original principal was invested, that is if $r = j$ and $t = m$, the formula (2) would reduce to the form

$$S = P(1 + j/m)^{mn},$$

which is the formula ordinarily employed in compound interest

expressed in terms of a nominal rate of interest payable several times a year.

3. *The Investment of a Single Sum. Three Rates of Interest.* It is certainly possible for the regular interest payments to be sufficiently large to enable the investor to invest them at a more favorable rate than he could realize at his savings bank, in which case he would be interested in the *amount* of his investment where three rates of interest would be involved. If the rate of interest realized on the original principal is j payable m times a year, the rate on these interest payments is r payable t times a year, and the rate on the second and all successive systems of interest payments is ρ payable θ times a year the amount at the end of n years, according to formula (2), is

$$S = P\{1 + \Sigma(j/m)(1 + rs_{\overline{x}|}^{(t)})\},$$

at rate ρ/θ, where the summation extends from $x = 0$ by intervals of $1/m$ to $x = n - 1/m$ inclusive. This expression reduces without difficulty to

$$(3) \qquad S = P\left\{1 + nj + \frac{jr\theta(s_{\overline{\theta|rt}} - mns_{\overline{\theta|m}})}{mt\rho\, s_{\overline{\theta|m}}\, s_{\overline{\theta|t}}}\right\}.$$

It is easily verified that when $\rho = r$ and $\theta = t$ formula (3) reduces to formula (2).

4. *The Amount of an Annuity. Two Rates of Interest.* We shall consider first the amount or total accumulations of the simplest form of an annuity of 1 per annum. As the amount of an annuity is simply the sum of the amounts of all of the various payments, the amount of the annuity, which is denoted by S_n, is

$$S_n = (1 + is_{\overline{n-1|}}) + (1 + is_{\overline{n-2|}}) + \cdots + (1 + is_{\overline{1|}}) + 1$$
$$= n + i(s_{\overline{1|}} + s_{\overline{2|}} + \cdots + s_{\overline{n-1|}}),$$

at rate r, which reduces easily to

$$(4) \qquad S_n = n + (i/r)(s_{\overline{n|}} - n)$$

at rate r. If $r = i$, formula (4) evidently reduces at once to $s_{\overline{n|}}$.

If the annuity of 1 per annum were payable p times a year, the interest at rate j on these payments were payable m times a year, and the interest at rate r on these interest payments were payable and compounded t times a year the amount of the annuity would be, by (2),

$$S_n = \Sigma(1/p)(1 + js_{\overline{x}|}^{(m)}),$$

at rate r/t, where the summation extends from $x = 0$ by intervals of $1/p$ to $x = n - 1/p$ inclusive. This expression reduces without difficulty to

$$S_n = n + \frac{jt(s_{\overline{tn}|} - nps_{\overline{t/p}|})}{pmr\, s_{\overline{t/m}|}\, s_{\overline{t/p}|}}. \tag{5}$$

at rate r/t.

If the annuity and all the interest payments were payable only once a year, that is if $p = m = t = 1$, formula (5) would reduce to formula (4). Similarly, if $j = r$ and $m = t$, formula (5) reduces to formula (A).

5. *The Amount of an Annuity. Three Rates of Interest.* If the annuity of 1 per annum were payable p times a year, the interest at rate j on these payments were payable m times a year, the interest at rate r on these interest payments were payable t times a year and the interest (ρ) on the second and all successive systems of interest payments were payable θ times a year the amount of the annuity, according to formula (3), would be

$$S_n = \Sigma \frac{1}{p}\left\{1 + xj + \frac{jr\theta(S_{\overline{\theta x}|} - mxS_{\overline{\theta/m}|})}{mt\rho\, s_{\overline{\theta/m}|}\, s_{\overline{\theta/t}|}}\right\},$$

at rate ρ/θ, where the summation extends from $x = 0$ by intervals of $1/p$ to $x = n - 1/p$ inclusive. This expression reduces then to

$$S_n = n + \frac{n(np-1)j}{2p}\left(1 - \frac{r\theta}{t\rho s_{\overline{\theta/t}|}}\right) + \frac{jr\theta^2(s_{\overline{\theta n}|} - nps_{\overline{\theta/p}|})}{pmt\rho^2 s_{\overline{\theta/m}|} s_{\overline{\theta/t}|} s_{\overline{\theta/p}|}}, \tag{6}$$

at rate ρ/θ.

It is easily verified that if $\rho = r$ and $\theta = t$, formula (6) reduces to formula (5). If $\rho = r = j$ and $\theta = t = m$, the formula reduces to the familiar $s_n^{(p)}$ at rate j/m.

6. *Present Values. Perpetuities.* The general formula for the present value of a single sum due at a future time where two or more rates of interest are involved is obtained by solving formula (3) for P. One must be careful, however, in problems involving several sums, to apply the formula thus obtained only when circumstances justify the application. For example, a formula so obtained should rarely be applied to compute the present value of an annuity. A little reflection will make it clear that the assumption that the same rate of interest will be realized on all interest income as on the principal may well be valid. For since the regular payment of the annuity is greater than the interest on the principal for the

same period, except in the case of a perpetuity, the one who provides the annuity needs in practice to withdraw only enough of the principal which supplemented by the interest will just provide the regular payment of the annuity. If the intervals between the payments of the annuity and the intervals of conversion of interest are the same, this procedure is equivalent to withdrawing the amount of the full payment of the annuity from the principal and then investing the interest at the same rate of interest at which the principal is invested. In any case the solution of the general problem of computing the present value of several sums due at various future times requires a careful analysis of the individual problem and can not be obtained in all cases by employing one general formula. The truth of this statement is further exemplified by the fact that it is apparently impossible to obtain such a formula which is concise and simple to apply, particularly in the case of an annuity.

The problem of finding the present value of a perpetuity whose payments are made at intervals of k years, where the interest payments are to be invested at a different rate of interest from that realized upon the principal, reduces simply to that of finding the principal whose interest payments will accumulate at the second rate to the successive payments of the perpetuity as they become due. If we denote such a principal by A and the successive payments by R, we may write

$$Ais_{\overline{k}|} = R,$$

$$A = \frac{R}{is_{\overline{k}|}}, \tag{7}$$

at rate r, which is the same formula as that ordinarily employed but is to be valued at another rate r (instead of i).

If the principal is invested at rate j payable m times a year and these interest payments and all further interest payments are invested at rate r payable t times a year the principal necessary to provide a perpetuity of R at intervals of k years is

$$A = \frac{R}{is_{\overline{k}|}^{(m)}}. \tag{8}$$

The formula corresponding to three rates of interest would be obtained by replacing $s_{\overline{k}|}^{(m)}$ by that portion of formula (3) included in the brackets minus unity. The formula would be valued, of course, at the third rate of interest.

7. *Valuation of Bonds. Two Rates of Interest.* We shall now derive formulas for valuating a bond which are based upon the assumption that the amortization factors will be invested at another rate of interest than that expected to be realized upon the principal of the bond.

We shall consider first the simplest type of a bond where the face value C is to be repaid in a single sum at the end of n years. If we denote the dividend rate offered in the bond offering by g, the rate to be realized by i, where both interests are payable but once a year, and the corresponding price by A, the amortization factor is $Cg - Ai$. If we treat the amortization factor as the annual payment into a sinking fund which is to accumulate at another rate of interest r and which is to account finally for the reduction in principal or $A - C$, we have

$$(Cg - Ai)s_{\overline{n}|} = A - C,$$

at rate r, where, in the case the bond is bought at a discount, the amortization factor and the reduction in principal are both negative. Solving for A, we find

$$A = C\,\frac{1 + gs_{\overline{n}|}}{1 + is_{\overline{n}|}},$$

at rate r. If we let $C = 1$, A becomes the price per dollar or $1 + k$, where k is the premium or discount per dollar. Solving for k, we obtain

$$k = \frac{g - i}{\frac{1}{s_{\overline{n}|}} + i}, \tag{9}$$

at rate r. If it should prove to be possible to invest the amortization factors at the rate planned to be realized on the whole bond, that is, if $r = i$, formula (9) reduces to the formula

$$k = a_{\overline{n}|}(g - i), \tag{B}$$

at rate i, since

$$\frac{1}{s_{\overline{n}|}} + i = \frac{1}{a_{\overline{n}|}},$$

at rate i. Formula (B) is the formula ordinarily employed to value a bond. If all the interests are payable m times a year, the formula for k is easily found to be

$$k = \frac{g - j}{\frac{m}{s_{\overline{mn}|}} + j}, \tag{10}$$

at rate r/m, where j is the nominal rate to be realized.

If the principal of a bond is to be repaid in installments it is easily shown that the premium or discount per dollar on the whole bond is simply the weighted arithmetic average of the premiums or discounts per dollar corresponding to the various installments treated as principals of distinct bonds where each premium or discount is weighted by the amount of the corresponding installment. For, if $C_1, C_2, \cdots, C_s$ are the values of the installments and $k_1, k_2, \cdots, k_s$ are the corresponding premiums or discounts per dollar found by one of the formulas derived above, the total price of the whole bond, say $C(1+k)$, where k is the premium or discount per dollar on the whole bond, is given by the formula

$$C(1+k) = C_1(1+k_1) + C_2(1+k_2) + \cdots + C_s(1+k_s).$$

But $C = C_1 + C_2 + \cdots + C_s$. Therefore

$$k = \frac{C_1k_1 + C_2k_2 + \cdots + C_sk_s}{C}. \tag{11}$$

Formulas (10) and (11) can then be employed to value any bond whether the principal is to be repaid in a single sum or in installments.

Formula (10) is especially valuable because it can be applied so easily to solve the inverse problem, namely, given the price of a bond to find the rate of interest which will be realized, since on clearing of fractions j occurs only to the first degree. Those who are familiar with the difficulties encountered in employing other formulas for this purpose will appreciate the advantage to be gained in employing formula (10). This application of formula (10) is so important that the solution of the formula for j is given as follows:

$$j = \frac{g - \dfrac{km}{s_{\overline{mn}|}}}{1+k}, \tag{12}$$

at rate r/m.

Incidentally, if the amortization factors are not invested at all, that is if $r = 0$, formula (12) reduces to the form

$$j = \frac{g - \dfrac{k}{n}}{1+k}. \tag{13}$$

DARTMOUTH COLLEGE,
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THE SPREAD OF NEWTONIAN AND LEIBNIZIAN NOTATIONS OF THE CALCULUS.

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(Read before the American Mathematical Society April 9, 1921.)

THE bitter controversy over the invention of the calculus is generally supposed to have eliminated the Leibnizian notation from Great Britain down to the time of Woodhouse, Peacock, Herschel and Babbage, and to have prevented the use of the Newtonian notation on the European continent. It is the purpose of this article to point out the extent to which this general impression is in need of revision and also to indicate the spread of these notations in America.

1. *Leibnizian Notation in Great Britain.* Several years before Newton permitted his theory of fluxions and his notation for fluxions and fluents to see the light of day in printed form, his friend, John Craig, used the notation of Leibniz, dp, dx, dy, in a book, the *Methodus Figurarum* published in 1685 in London. Craig employed the Leibnizian notation again in 1693 in another booklet, the *Tractatus Mathematicus*, as well as in articles printed in the PHILOSOPHICAL TRANSACTIONL OF LONDON for the years 1701, 1703, 1704, 1708; his article of 1703 contains the sign of integration $\int$. In the TRANSACTIONS for 1704–05, an article by John Bernoulli makes extensive use of the Leibnizian signs. In 1716 the English physician and philosophical writer, George Cheyne, brought out in London the *Philosophical Principles of Religion*, Part II, which contains a chapter from the pen of John Craig on a discussion of zero and infinity, dated September 23, 1713. In this chapter Craig uses Leibniz's symbols for differentiation and integration. But in 1718 Craig made a complete change. In that year he issued a book, *De Calculo Fluentium*, in which he switches over to the exclusive use of the Newtonian notation. Evidently this change is a result of the controversy then raging between the supporters of Newton and Leibniz.

In volume 23 of the PHILOSOPHICAL TRANSACTIONS OF LONDON, for the years 1702 and 1703, De Moivre uses dots for fluxions, but in integration he uses the Leibnizian sign $\int$. Thus, on page 1125, De Moivre writes "adeoq; $\dot{q} = \frac{3}{2}dv^2\dot{v} - \frac{3}{2}dv^2\dot{y}$, igitur $q = \frac{1}{2}dv^3 - \int \frac{3}{2}dv^2\dot{y}$. Ergo ad hoc perventum est ut

fluentum quantitatem inveniamus cujus fluxio est $\frac{2}{3}dv^2\dot{y}$." Evidently his partisanship favoring Newton did not at this time prevent his resorting to the convenience of using the Leibnizian symbol of integration.

Even John Keill, who fought so prominently and unskillfully on the side of Newton, employed a similar mixed notation. In a paper, dated November 24, 1713, and printed in the PHILOSOPHICAL TRANSACTIONS for 1714–16, he adopts the symbolism "$\int \phi\dot{x}$". A mixture of Continental and British symbols is found in the writings of E. Waring and much later in Olinthus Gregory's *Treatise of Mechanics* (3rd Ed., London, 1815) where on page 158 there is given "$\int dy\sqrt{(\dot{x}^2+\dot{y}^2)}$." In times still more recent the Newtonian dot has been used to advantage, by the side of Leibnizian symbols, in the treatment of mechanics and the algebras of vectors.*

Returning to the eighteenth century, we find Benjamin Robins† writing the Leibnizian signs in a review of a publication of Leonhard Euler. Joseph Fenn, an Irish writer, little known, who had studied on the Continent, and was at one time professor of philosophy in the University of Nantes, issued at Dublin soon after 1768, a *History of Mathematics*. In it Fenn has occasion to use the calculus, and employs the Leibnizian notation. In a *Plan of the Instructions given in the Drawing School established by the Dublin Society* (p. LXXXIX of the above volume) he discusses the tides, and, in finding the greatest height of the tides, he uses the calculus and the notation of Leibniz. He uses them again in his *Second Volume of the Instructions given in the Drawing School established by the Dublin Society*, Dublin, 1772. He is friendly to Newton, uses the terms "fluxion" and "fluent," but never uses Newton's notation. He is perhaps the last eighteenth century writer in Great Britain who used the symbolism of Leibniz.

From our data it is evident that the Leibnizian notation was the earliest calculus notation in England which appeared in print, that in rare instances it was used by certain Englishmen, but that it vanished almost completely from British soil in the latter part of the eighteenth century.

* See, for example, P. G. Tait's *Elementary Treatise on Quaternions*, 2nd Ed., 1873, Chapter X; T. Hayashi in this BULLETIN, vol. 26, 1919, p. 74.

† Benjamin Robins, *Remarks on Mr. Euler's Treatise on Motion, Dr. Smith's Compleat System of Opticks, and Dr. Jurin's Essay upon Distinct and Indistinct Vision*, London, 1739.

2. *Newtonian Notation in Continental Europe.* That Newton's notation would be reproduced in editions of his works printed on the European continent is to be expected. Thus the fluxional symbols appear in J. Castillon's edition of Newton's mathematical works, brought out at Lausanne and Geneva in 1744, and in the appendix to the second volume of Castillon's edition of Newton's *Arithmetica Universalis*, Amsterdam, 1761. John Muller, a German by birth, who was appointed master of the Royal Military Academy at Woolwich and became "the scholastic father of all the great engineers this country (England) employed for forty years," brought out in 1736 in London a book entitled *A Mathematical Treatise: Containing a System of Conic-Sections; with the Doctrine of Fluxions and Fluents.* Muller translated his own book into French and had it published at Paris in 1760. The French as well as the English editions used the Newtonian symbols. An article from the pen of David Gregory at Oxford, after its appearance in the PHILOSOPHICAL TRANSACTIONS OF LONDON, was republished in 1697 in the ACTA ERUDITORUM of Leipzig, the British fluxional symbols being faithfully reproduced. This procedure is no more unusual than was the printing of John Bernoulli's article in the PHILOSOPHICAL TRANSACTIONS of 1704–05. The fluxional notation is given in full in the article "Fluxions" in Alexandre Savérien's *Dictionnaire Universel de Mathématique et de Physique*, Paris, 1753. We proceed to more startling disclosures.

After the middle of the eighteenth century when feelings of resentment ran high between the adherents of Leibniz and their opponents, it is extraordinary that a mathematical journal should have been published on the European continent for fifteen years which often uses the Newtonian notation, but never uses the Leibnizian. It is more surprising still, that no reference is made to this fact in any history of mathematics, though many volumes have been written which deal with various phases in the evolution of the calculus. The journal in question is a monthly, published at Amsterdam from 1754 to 1769. It appeared under the title, *Maandelykse Mathematische Liefhebbery* (Monthly Mathematical Recreations). Each issue contained about forty pages. It is mentioned in Cantor's history,* but otherwise has not attracted

* M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. IV, Leipzig, 1908, p. 70.

the attention of writers on the general history of mathematics. This neglect is due, probably, to its appearing in the Dutch language and devoting itself to the solution of elementary problems on algebra, arithmetic and geometry, and to the publication of school items of only local interest. It contains problems on maxima and minima, solved with the aid of fluxions dressed in the familiar garb of the Newtonian dots. The first twelve volumes of the journal give altogether forty-eight or more such problems; the last few volumes contain no fluxions. About seventeen different writers contributed fluxional solutions. Some writers give only their initials; none of them are men of note. It was a time when the Netherlands did not have great mathematicians. Huygens had died in 1695; Hudde in 1704; Nieuwentijt in 1718. Among the contributors using fluxions were G. Centen, J. Schoen, J. Kok, J. T. Kooyman, F. Kooyman, A. Vryer, J. Bouman, D. Bocx. The two most noted where P. Halcke, and Jakob Oostwoud. The latter was editor of the monthly during the first twelve years. Oostwoud was a teacher of mathematics in Oost-Zaandam near Amsterdam, who in 1766 became member of the Hamburg Mathematical Society.* Later he collected and published the problems solved in his journal in three separate volumes.†

3. *Calculus Notations in the United States.* In the United States the early influence was predominantly English. Wherever higher analysis received attention, it was in the form of the fluxions of Newton. Before 1766 occasional studies of conic sections and fluxions were undertaken at Yale‡ under the inspiring leadership of President Clap. Perhaps as early as 1795, and again in 1802, Jared Mansfield published at New Haven, Conn., a volume of *Essays, Mathematical and Physical,* which includes a part on "fluxionary analysis." In a footnote, on page 207, he states that the fluxions of x, y, z are

* Bierens de Haan in *Festschrift, herausgegeben von der Mathematischen Gesellschaft in Hamburg, anlässlich ihres 200 jährigen Jubelfestes 1890.* Erster Teil, Leipzig, 1890, p. 79.

† Bierens de Haan, op. cit., p. 80, where the titles of the three publications are given in full. See also de Haan's *Bouwstoffen voor de Geschiedenis der Wis-en Natuurkundige Wetenschappen in de Nederlanden,* 1878, p. 76–85, reprinted from *Verslagen en Mededeelingen der Kon. Akademie van Wetenschappen,* Afd. Natuurk., 2e Reeks, Deel VIII, IX, X en XII.

‡ *Yale College; a Sketch of its History,* by William L. Kingsley, Vol. II, pp. 497–498.

usually denoted "with a point over them, but we have here denoted these by a point somewhat removed to the right hand," thus $x^{\cdot}$. A similar variation occurs in Robert Woodhouse's *Principles of Analytical Calculations*, Cambridge (England), 1803, p. XXVII, where we read, "Again $\overset{\cdot\cdot\cdot}{xy}$ or $(xy)^{\cdot\cdot\cdot}$ is not so convenient as $d^3(xy)$."

In 1801 Samuel Webber, then professor of mathematics and later president of Harvard College, published his *Mathematics compiled from the Best Authors*. In the second volume he touched upon fluxions and used the Newtonian dots. This notation occurred also in the TRANSACTIONS OF THE AMERICAN PHILOSOPHICAL SOCIETY, of Philadelphia, in an article by Joseph Clay who wrote in 1802 on the "Figure of the Earth." It is found also in the second volume of Charles Hutton's *Course of Mathematics*, American editions of which appeared in 1812, 1816(?), 1818, 1822, 1828 and 1831.

An American edition of S. Vince's *Principles of Fluxions* appeared in Philadelphia in 1812. That early attention was given to the study of fluxions at Harvard College is shown by the fact that in the interval 1796–1817 there were deposited in the college library twenty-one mathematical theses which indicate by their titles the use of fluxions.* These were written by members of Junior and Senior classes. The last thesis referring in the title to fluxions is for the year 1832.

At West Point, during the first few years of its existence, neither fluxions nor calculus received much attention. As late as 1816 it is stated in the West Point curriculum that fluxions were "to be taught at the option of professor and student." In 1817, Claude Crozet, trained at the Polytechnic School in Paris, became teacher of engineering. A few times, at least, he used in print the Newtonian notation, as, for instance, in the solution, written in French, of a problem which he published in the PORTICO, of Baltimore, in 1817. Robert Adrain, later professor in Columbia College and also at the University of Pennsylvania, used the English notation in his earlier writings, for example, in the third volume of the PORTICO, but in Nash's LADIES AND GENTLEMEN'S DAIRY, No. II, published in New York in 1820, he employs the dx.

A slight deviation from the Newtonian forms is found in

* Justin Winsor, *Bibliographical Contributions*, No. 32. Issued by the Library of Harvard University, Cambridge, 1888.

articles on the theory of fluxions, contributed by Elizur Wright of Tallmadge, Ohio, to the AMERICAN JOURNAL OF SCIENCE AND ARTS, for the years 1828, 1833, 1834. Like Mansfield, he does not place the dot exactly over the variable, but a little to the right, where ordinarily exponents are put. Thus, the fluxion of x is $x^{\cdot}$; the first, second and third fluxions of x^3 are, respectively, $3x^2x^{\cdot}$; $6x\ x^{\cdot 2}$, $6x^{\cdot 3}$. After 1830 the Newtonian fluxional system is rarely encountered in the United States.

In the AMERICAN JOURNAL OF SCIENCE AND ARTS, the earliest paper giving the Leibnizian symbols was prepared by Professor A. M. Fisher of Yale College. It is printed in Volume 5 (1822) and is dated August, 1818.

The earliest articles in the MEMOIRS OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES which contain the "d-istic" signs bear the date of 1818 and were from the pens of F. T. Schubert and Nathaniel Bowditch.* The latter came to know the calculus on long sea voyages during the years 1795 to 1804, when he studied Lacroix's Calculus. Bowditch began his translation of Laplace's *Méchanique Céleste* in 1814. In the MEMOIRS mentioned above, Theodore Strong of Rutgers College began in 1829 to publish a series of papers containing solutions of problems, involving the Leibnizian symbolism.

The articles contributed by Fisher, Schubert, Bowditch and Strong were isolated papers which were not accessible to the mass of students in mathematics. The publication which placed the Leibnizian calculus and notation within reach of all and which marks the time of the real beginning of their general use in the United States, was the translation from the French of Bezout's *First Principles of the Differential and Integral Calculus*, made by John Farrar of Harvard University in 1824. Between 1824 and 1831 five mathematical theses were prepared at Harvard which contain in their titles the words "Differential Calculus" or "Integral Calculus."† After 1824 French influences dominated all instruction in higher analysis at Harvard.

From what we have stated it appears that the movement toward the introduction of continental analysis began in America about ten years later than it did in England.

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* MEMOIRS OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES, Vol. 4, Part I, Cambridge, Mass., 1818, pp. 5, 47.

† Justin Winsor, op. cit.

GROUP THEORY REVIEWS

IN THE

JAHRBUCH ÜBER DIE FORTSCHRITTE DER MATHEMATIK

BY PROFESSOR G. A. MILLER.

A few erroneous statements found in a recent volume of the JAHRBUCH ÜBER DIE FORTSCHRITTE DER MATHEMATIK and relating to the theory of groups led the present writer to make a brief survey of the other volumes of this series for the purpose of determining the reliability of the reviews relating to this particular subject. As these reviews are so widely read and so frequently referred to, it seems desirable to note here a few of the instances of inaccurate or misleading statements found therein, since some of these instances may be instructive and since such a notice may tend to prevent a repetition of these particular errors. The fact that a few of the many reviewers failed to maintain the high standards of this classic series should not be surprising, and a knowledge of their shortcomings can only increase the usefulness of this series, which has had no serious competitor, since its inauguration about half a century ago, in its field of providing critical reviews of the entire current mathematical literature relating to important advances.

The erroneous statements mentioned above are found on page 164 of volume 44 (1918) and relate to the possible sets, or systems, of independent generators of a finite group whose order is a power of a prime. It is there stated that the number of the operators in such a set is equal to the index of the commutator subgroup. In the following sentence the equally incorrect statement is made that a system of independent generators of such a group can be obtained by taking one operator from each of the co-sets with respect to the commutator subgroup. According to these statements the cyclic group of order p^m, p being a prime number, would have a set of p^m independent generators, since the commutator subgroup of this group is the identity. On the contrary, it is well known that only one operator can appear in such a set of generators.

These statements are so obviously incorrect that one might be inclined to attribute them to a harmless and amusing over-

sight if they were not followed by a sentence which is not only incorrect but also practically meaningless. In that sentence, it is stated that the author in question considered the series of commutator subgroups and made assertions about their minimal order. As a matter of fact minimal orders of the *centrals* of these commutator subgroups were considered in the article under review. From what has been noted, it is evident not only that the reader of this particular review will get very little knowledge therefrom regarding the significance of the article in question, but that he is also in danger of being led to regard a theorem as true which he will find later to be false.

It may be of interest to note in this connection that each one of the published volumes of the JAHRBUCH contains reviews of articles on group theory and that more than thirty different men wrote these reviews. Most of these men published only a few such reviews. Among the early reviewers along this line E. Netto stands out most prominently for his long service and the number of articles he reviewed. In particular, Netto reviewed most of C. Jordan's articles on this subject as well as his now classic *Traité des Substitutions*, 1870. These reviews of Jordan's work were not always fair. For instance, in the review of this *Traité*, Netto said, "there are still many gaps to be filled, much diffuseness to be removed, and many proofs to be corrected."

The fact that Jordan was not pleased with this review is clear from a note published in volume 79 of CRELLE (p. 258), where Jordan refers to the first publication of Netto in support of the criticism quoted above, and says that Netto was not entirely successful in his effort. Jordan adds that he could not hope that his *Traité des Substitutions* was free from inexact statements in view of its great extent and of the novelty and difficulty of the subject treated, but that he was convinced that there were few such statements; he himself had found two which he corrected promptly and he would be greatly obliged to those who would point out others.

In a review which appeared on page 41 of the first volume of the JAHRBUCH, Netto makes an incorrect statement relating to a fundamental question, and hence it may be worth noting here. After defining the term *isomorphic group* in accord with our present definition of *simply isomorphic group*, Netto considers n rational functions $F_1, F_2, \cdots, F_n$ of the m variables x_1,

$x_2, \cdots, x_m$, which constitute a complete set of conjugates under a substitution group G on these m variables. He observes correctly that the substitutions of G transform F_1, $F_2, \cdots, F_n$ according to a transitive substitution group, but he adds incorrectly that this transitive group is isomorphic (simply isomorphic) with G. It is now well known that a necessary and sufficient condition that the transitive group on the F's be simply isomorphic with G is that the subgroup composed of all the substitutions of G which transform into itself one of these F's should be non-invariant under G and should not include any invariant subgroup of G besides the identity.

An article of unusual importance in the development of group theory is the one in which A. Capelli introduced the concept of general isomorphisms between two groups. It is therefore of interest to note here several inaccuracies which appear in the review of this article, published on page 106 of volume 10 of the JAHRBUCH. Near the beginning of this review the term subgroups is used to denote co-sets, and the definition of general isomorphisms is obscured thereby. Near the end of the review the statement that a transitive group of order p and of degree $n > p$ cannot be primitive is found. It was known even at the time of A. L. Cauchy that the order of a transitive group is always divisible by its degree and that therefore there can be no transitive group of order p and of degree $n > p$. In the same review, theorems relating to Sylow subgroups, which were well known at a much earlier date, were spoken of as if they were new.

On page 116 of volume 16 it is stated that in groups of order $n = p^\alpha \cdot q^\beta \cdots$, where $p, q, \cdots$ are prime numbers, which contain only one subgroup of each of the orders p^α, $q^\beta, \cdots$, all the substitutions are commutative. On the following page, it is said that these groups are the only ones in which every subgroup is invariant. As a matter of fact the non-abelian groups in which every subgroup is invariant were investigated much later and were named Hamiltonian groups by R. Dedekind. It is true that each of these groups contains only one Sylow subgroup for every prime which divides the order of the group, but this is not a sufficient condition for a Hamiltonian group. It is fortunate that the statement just noted was not accepted as true since otherwise the interesting later investigations relating to Hamiltonian groups would probably not have been made.

It is interesting to note that some of the most important articles were reviewed very briefly. For instance, Netto devoted only eight lines to the review of the important article by L. Sylow which appeared in volume 5 of the MATHEMATISCHE ANNALEN. A special case of a fundamental result contained in this article but which Netto failed to mention was later credited to Netto in a review written by F. Meyer and published on page 139 of volume 20 of the JAHRBUCH. The latter review is such a mixture of meaningless statements and trivialities that it would be almost amusing if it did not appear in a work which maintains, on the whole, very high standards.

In view of the fact that most of the reviewers for the JAHRBUCH have been more familiar with the work of German authors than with that of authors of other lands, it is only natural that one sometimes finds undue credit given to the former. This does not imply that these reviewers were conscious of any unfairness in giving credit. An interesting example of this apparently unconscious bias is found in the review of an article by the present writer, on page 98 of volume 27. It is there stated that the theorem that every group of order p^α, p being a prime number and $\alpha > 3$, contains an abelian subgroup of order p^3, is a consequence of theorems due to Hölder and Frobenius. As a matter of fact Hölder needed this theorem but failed to find it when he was determining the groups of order p^4, and he devoted more than 30 pages of the MATHEMATISCHE ANNALEN (vol. 43) to a discussion of properties of these groups. When the present writer communicated this theorem to Hölder, the latter replied that after receiving this information he at once went over his groups of order p^4 and verified that each of them actually contains an abelian subgroup of order p^3. He realized that the use of this theorem would have greatly simplified his discussions. The theorem might be said to be a consequence of developments due to Sylow rather than of work done by Hölder and Frobenius.

UNIVERSITY OF ILLINOIS,
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RECENT BOOKS ON VECTOR ANALYSIS.

Einführung in die Vektoranalysis, mit Anwendungen auf die mathematische Physik. By R. Gans. 4th edition. Leipzig and Berlin, Teubner, 1921. 118 pp.

Elements of Vector Algebra. By L. Silberstein. New York, Longmans, Green, and Company, 1919. 42 pp.

Vektoranalysis. By C. Runge. Vol. I. *Die Vektoranalysis des dreidimensionalen Raumes.* Leipzig, Hirzel, 1919. viii + 195 pp.

Précis de Calcul géométrique. By R. Leveugle. Preface by H. Fehr. Paris, Gauthier-Villars, 1920. lvi + 400 pp.

The first of these books is for the most part a reprint of the third edition, which has been reviewed in this BULLETIN (vol. 21 (1915), p. 360). Some small condensation of results has been accomplished by making deductions by purely vector methods. The author still clings to the rather common idea that a vector is a certain triple on a certain set of coordinate axes, so that his development is rather that of a shorthand than of a study of expressions which are not dependent upon axes.

The second book is a quite brief exposition of the bare fundamentals of vector algebra, the notation being that of Heaviside, which Silberstein has used before. It is designed primarily to satisfy the needs of those studying geometric optics. However the author goes so far as to mention a linear vector operator and the notion of dyad. The exposition is very simple and could be followed by high school students who had had trigonometry.

The third of the books mentioned is an elementary exposition of vectors from the standpoint of Grassmann. The author uses Gibbs' notation, abandoning the usual German notation of parentheses and brackets. This is a step in the right direction, though matters would be still further improved by using the Hamiltonian notation complete. The development of the *tensor* (dyadic)* is after Gibbs. The treatment is entirely

* *Tensor* as here used means dyadic, though generally it means *symmetric dyadic, self-transverse matrix.*

for ordinary space, consideration of four-dimensional and higher space, useful for relativity, being deferred to the succeeding volume.

The first chapter is on vectors and plane magnitudes or quantities, and defines the various kinds of products. The second chapter is on differentiation and integration. The third is on tensors or dyadics. Simple applications are introduced where needed, the development reaching the usual elementary theorems on divergence, curl, Stokes' theorem and Green's theorem. The author uses the vector directly and only occasionally reverts to vectors as triples. There is not much attempt to elaborate formulas.

The fourth book is all that the title implies, and as a manual for working mathematicians and students is quite complete. It was written during the captivity of the author. There is a carefully worked out plan of exposition, from general notions on complex numbers to applications to electricity. In a preface of fifteen pages the general plan is explained with reasons for the eclectic notation. The sixteen chapters are analyzed, article by article, in a table of contents, making the work very accessible to the reader who wishes to consult the book. This table occupies twenty-six pages. An introduction of four pages examines the idea of quantity, and gives certain historical references of use. We shall give some idea of the scope of the work.

The first chapter is fundamental. In it the author starts with the notion of *qualitative* number as a given foundation, defines units, and takes as primary notion that of number considered to be the sum of numerical multiples of the units. The totality of numbers thus formed for a given set of units constitute the *domain* of the units. Domains are identical when every number (hypernumber) in either is also in the other. The *degree* of the domain is the number of linearly independent units necessary to express every number of the domain. Units are not unique, in the sense that any system of numbers of the domain which are linearly independent and in number as many as the degree of the domain, will equally define the domain. The divergence of Hamiltonian developments from Grassmannian developments comes with the idea of the product of two hypernumbers. In the former a product is always in the domain of the total set of units; in the latter a product is a hypernumber of a different grade and not in the

domain. This very great distinction is usually omitted, overlooked, or not regarded as existing, in many treatments of vectors. It is, however, vital.

In the second chapter are defined vector, parallelogram (bivector) and *spath* (trivector). Addition of points and the barycentric calculus are defined, and the equations that result are deduced. The bipoint or glissant vector and the tripoint or glissant bivector are defined. The third chapter deals with combinatorial or alternating multiplication, progressive and regressive, exterior and interior. The fourth chapter applies these notions to geometry, kinematics, and statics. The notions are essentially those of Grassmann.

Chapters five and six consider the product of vectors and the scalar and vector parts of the products. These notions are essentially quaternionic. The fundamental formulas are derived from the postulated laws of multiplication of the unit vectors i, j, k. Chapter seven deals with the differentiation of vectors.

Chapter eight applies all the foregoing developments to geometry, kinematics, twisted curves, and curved surfaces. In each case the fundamental theorems are proved and the necessary useful formulas are deduced.

Chapter nine develops in some detail the dyadic or linear vector operator. The notation of Gibbs is introduced in part. The invariants, scalar and vector, are found, and important special cases are studied. An application is made to quadrics.

Chapters ten and eleven consider the very important *nabla* of Hamilton, and the various resulting differential expressions. In chapter twelve multiple integrals with the theorems of Stokes and Green, and a study of vector fields, are to be found. These three chapters lay the foundations for mathematical physics. In the remaining chapters the theory of the potential, movement of a solid, elasticity, and electromagnetism, are treated.

The book is amply supplied with examples and exercises. The typography is up to the usual excellence of the publications of the firm, and few errors occur. As a satisfactory beginning book for the various forms of vector calculus it is to be highly recommended.

JAMES BYRNIE SHAW.

SHORTER NOTICES.

Oeuvres de G. H. Halphen. Publiées par les soins de C. Jordan, H. Poincaré, E. Picard, avec la collaboration de E. Vessiot. Vol. I, 1916, xliv + 570 pp. Vol. II, 1918, vii + 560 pp. Paris, Gauthier-Villars.

Picard in his *Notice sur la vie et les travaux de Georges-Henri Halphen* and Poincaré after him in his *Notice sur Halphen* urged that one could distinguish among the mathematicians of thirty years ago two well-marked and opposing tendencies of thought separating mathematical labors into two distinct categories and the mathematicians themselves, perhaps less definitely, into two classes. Those of the one class are preoccupied principally with enlarging the field of known notions; the others prefer to devote their energies primarily to penetrating more deeply into notions which already have been analyzed and elaborated. This distinction of classes is just as valid today as it was when Picard and Poincaré insisted upon it in 1890; and it is likely to remain so as long as mathematics develops along the lines already marked out.

Those who are most concerned with extending the frontiers of science often find it necessary to leave their new ideas without much elaboration and to proceed to a general account of the lay of the field, so to speak, in order to obtain at once a comprehensive view. In such investigations many questions will be raised and not answered, it is very difficult to stand always clear of errors in detail, lines of investigation which deserve to be followed up must at most be only indicated, and many promising thoughts must be dismissed altogether.

The mathematicians of the second class, of whom Halphen is one of the greater, are more intimately concerned with a desire to give to their work a character of absolute perfection. An error with respect to even the smallest detail becomes a matter of acute pain. Whatever these mathematicians touch they wish to achieve to the point of leaving unanswered no question which their investigation raises. They seek nothing less than to put their thought into a form of absolute perfection and beauty.

These two directions of mathematical thought are observed also in the different branches of the science. One can probably

still say (as Picard said in 1890) that in a general way the tendency to comprehensive investigation and broad characterization is found more often in matters relating to the theory of functions, and the tendency to deep penetration in the works on modern algebra and analytic geometry.

The work of Halphen is marked principally by the tendency to deep penetration. He devoted himself for the most part to algebra and geometry, fields apparently best suited to such a temperament. He analyzed the difficult problems attacked to the point of complete and definitive results. Everywhere his work is marked by the effort to leave not a single thing unachieved.

After the researches of de Jonquières and Chasles there was a lively investigation of algebraic systems of conics depending on a single parameter. Chasles had found, by a sort of induction, a general law giving the number of conics satisfying a given condition. This number was composed of a sum of two terms, each of them being a product of two factors one of which depended on the system and the other on the condition. Halphen, simultaneously with several other mathematicians, sought a proof of the law of Chasles; and he believed that he had found one. Later he perceived an error in his argument and consequently took up again a study of the question. After a long investigation his labors were rewarded through the discovery of a complete solution of the problem by means of a method of great originality. The result is distinguished by a characteristic deep penetration.

Halphen's masterpiece, according to Poincaré, is his memoir on twisted algebraic curves, crowned in 1881 by the Berlin Academy. This theory is contrasted in a remarkable manner with that of plane curves. For the latter a single number, the degree, suffices for a complete classification. Twisted algebraic curves do not possess a like property. To give a single number is not enough to classify them. Moreover, one cannot find a system of integers enjoying with respect to twisted curves a rôle analogous to that of the degree for plane curves. Halphen obtains a satisfactory solution of the problem; but, since the solution is not susceptible of an analytical expression, its character is likely to escape the superficial reader. The results bring to light certain curious and unnoticed properties which are brought out clearly by Poincaré (vol. I, p. xxxi).

Among the other larger subjects investigated by Halphen

may be mentioned the following: singular points of algebraic curves, differential equations and invariants, elliptic functions, theory of numbers, and theory of series.

For a brief sketch of the life of Halphen, with some remarks on the character of his work, the reader may consult the *Notice* by Picard (vol. I, pp. vii–xvi). In the *Notice* by Poincaré (vol. I, pp. xvii–xliii) we have an excellent systematic analysis of his mathematical contributions. Besides this there is the *Notice* by Halphen himself (vol. I, pp. 1–47) in which his own contributions were analyzed on the occasion of his candidacy before the Paris Academy of Sciences in 1885, about four years before his death. These excellent brief accounts of his work relieve the reviewer of the duty of making an analysis of the separate memoirs. The entire works (with the exception, apparently, of the *Traité des Fonctions Elliptiques*) are to be included in four volumes, of which the third is announced as in press and the fourth in preparation.

R. D. CARMICHAEL.

A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse. By Florian Cajori. Chicago, The Open Court Publishing Company, 1919, pp. viii + 299.

This work appears as number five in the Open Court Series of Classics of Science and Philosophy, a series which should meet with all the encouragement and support that American scholars can give in these times, when the question of the publication of such works is so critical. That such encouragement and support is justified may be seen from an examination of this latest production of Professor Cajori's pen, for he has here given to scholars one of the best of his various studies in the history of mathematics.

The work consists of twelve chapters under substantially the following titles: I. Newton; II. Printed books and articles on fluxions before 1734; III. Berkeley's *Analyst;* IV. Jurin's controversy with Robins and Pemberton; V. Textbooks immediately following Berkeley; VI. Maclaurin's *Fluxions* (1742); VII. Textbooks of the middle of the eighteenth century; VIII. Robert Heath and the controversy in his time; IX. Abortive attempts at arithmetization; X. Later works on fluxions; XI. Criticisms under the influence of

French writers; XII. Merits and defects of the early fluxional conceptions. The work has a good index which students will find helpful in the matter of ready reference.

The two features of the work that will appeal to the student as of paramount importance are the selection and arrangement of original material and the translations which accompany the Latin texts. For good or ill, the ability of American scholars to read Latin easily has departed; and although the original Latin forms are desirable for reference, either a translation or an explanation of some kind has now come to be a desideratum, if not a necessity. Professor Cajori has placed students under great obligation for searching out with care the passages in Wallis, Newton, and others who played leading parts in the perfecting of the calculus, and in presenting these passages in a manner that renders them accessible to all who care to consult them. Here the reader will find the early symbolism, the definitions, and the methods of approach that are necessary to an understanding of the numerous controversies that agitated mathematical England, with respect to the calculus, for more than a century.

In the first chapter Professor Cajori has given numerous excerpts from such works of Newton as appeared before 1734, the date of Bishop Berkeley's attack upon the whole theory of fluxions. These are chiefly from the *Principia* (1687), from Newton's *Quadrature of Curves*, which Wallis first published in his Algebra (1693), from the *Commercium Epistolicum* (1717) and from certain letters and manuscripts. These extracts show that Newton made use of infinitesimals as early as 1665, that he used the dot notation in the same year, that he first used the word "fluxion" (at least in print) in 1687, that he used the idea of limits in 1687, and that he finally placed his theory on a thoroughly logical basis in the 1704 edition of his *Quadrature of Curves*. The summary is a particularly lucid one of the steps taken by Newton in his development of the calculus.

The literature upon the subject which appeared in Great Britain between 1686, when John Craig published his *Methodus Figurarum . . . quadraturas determinandi*, and 1734, when Berkeley's *Analyst* appeared, is next considered. Professor Cajori shows that the influence of Leibniz was much greater in England than is generally suspected, and shows that the clashing of the two different methods of attack, the New-

tonian and the Leibnizian, gave to Berkeley an excellent opportunity for "the most spectacular event of the century in the history of British mathematics." Mathematicians are generally aware of the fact that this attack was made, but the details are rarely known. Berkeley's *Analyst* is not a common work, although it is occasionally offered by dealers in the classics of the eighteenth century. For this reason the careful summary here given will be welcome to those who care for the genesis and early status of great movements in the field of their favorite science. The extracts from Berkeley, for example, have been carefully copied, although the author has taken certain justifiable liberties in the matter of punctuation, spelling, and capitalization.

A further service rendered by Professor Cajori is seen in his chapters on the books on the fluxional calculus that appeared after Berkeley's attack; many of them, no doubt, being the natural result of such criticism, the authors appearing as champions for a cause that seemed to them worthy of strong defence. For the interesting details of this defence, and for the subsequent fortunes of the fluxional type of analysis, however, the reader should consult the work itself.

The mathematical world has often been in debt to Professor Cajori for his detailed studies, but never more so than in this case.

DAVID EUGENE SMITH.

Cours de Cinématique théorique. By H. Lacaze. Paris, Gauthier-Villars, 1920. 138 pp.

This text is for the use of students of the lycée and the government schools. The first part of the book covers 56 pages, the complementary part 82 pages. In the first part are five chapters entitled, respectively, vectors, kinematics of a point, movement of a solid and distribution of velocities, composition of accelerations, and displacement of a vector in a plane. The complementary part has five corresponding chapters. Vectors are represented throughout as triples, no real vector notation entering the book. The first part is elementary, the complementary part is more advanced. Explanations are brief but sufficient, and a few exercises are introduced which help out the book for student use. It is clear, well-printed, and ample for a beginning course.

JAMES BYRNIE SHAW.

Some famous problems of the theory of numbers and in particular Waring's problem. An inaugural lecture delivered before the University of Oxford by G. H. Hardy. Oxford, Clarendon Press, 1920. 8vo. 34 pp.

The particular problems with which this lecture is concerned belong to the additive theory of numbers. The general problem of the latter is stated by Hardy as follows: "Suppose that n is any positive integer, and $\alpha_1, \alpha_2, \alpha_3, \cdots$ positive integers of some special kind, squares, for example, or cubes, or perfect kth powers, or primes. We consider all possible expressions of n in the form $n = \alpha_1 + \alpha_2 + \cdots + \alpha_s$, where s may be fixed or unrestricted, the α's may or may not be necessarily distinct, and order may or may not be relevant, according to the particular problem on which we are engaged. We denote by $r(n)$ the number of representations which satisfy the conditions of the problem. Then *what can we say about* $r(n)$? Can we find an exact formula for $r(n)$, or an approximate formula valid for large values of n? In particular, is $r(n)$ *always positive*? Is it always possible, that is to say, to find at least *one* representation of n of the type required? Or, if this is not so, is it at any rate always possible when n is sufficiently large?"

The number $p(n)$ of unrestricted partitions of n into positive integral summands has been studied by many authors; the principal result of the investigation of this function by Hardy and Ramanujan has been the discovery of an approximate formula for $p(n)$ which enables them to approximate to $p(n)$ with an accuracy which is almost uncanny. Of $p(200)$, for example, the value 3,972,999,029,388 is obtained with an (additive) error of .004 by employing eight terms of their series; and the result has been verified by MacMahon, without the use of their formula, by a direct computation which occupied over a month.

The principal object of the lecture is a discussion of the problem of Waring of determining the number of representations of an integer n as a sum of s positive kth powers of integers and particularly of the (more usual) restricted form of this problem in which one seeks to show that for fixed k there exists a finite s_k independent of n such that every integer n has at least one representation as a sum of s_k non-negative kth powers. In connection with this problem there are two functions of fundamental importance, whose existence has

been proved in recent years; they may be defined as follows: The number $g(k)$ is defined to be the least number for which it is true that every positive integer is the sum of $g(k)$ non-negative kth powers of integers; the number $G(k)$ is defined to be the least number for which it is true that every positive integer from a certain point onwards is the sum of $G(k)$ non-negative kth powers of integers. The existence of either of the functions $G(k)$ and $g(k)$ obviously implies that of the other. The existence of $g(k)$ was first proved by Hilbert in 1909, after an interval of 139* years from the time of its enunciation by Waring,† who gave the theorem without proof.

For a long time it has been known that $g(2) = G(2) = 4$. In 1859 Liouville proved that $g(4)$ exists and does not exceed 53; it was shown by Wieferich in 1909 that $g(4) \leqq 37$, the most that is known at present. The number $79 = 4 \cdot 2^4 + 15 \cdot 1^4$ needs 19 biquadrates, and no number is known which needs more. There is still therefore a wide margin of uncertainty as to the actual value of $g(4)$. The existence of $g(3)$ was first established in 1895 when Maillet proved that $g(3) \leqq 17$; Wieferich proved in 1909 that $g(3) \leqq 9$. As 23 and 239 require exactly 9 cubes, the value of $g(3)$ is exactly 9. [Hardy remarks that it is "no doubt true" that 23 and 239 are the only integers requiring 9 cubes for their expression.] In 1909 Landau proved the "singularly beautiful theorem" that the number of integers requiring 9 cubes each for its expression is finite. It was in view of this fact that the number $G(k)$ was introduced. It is known that $4 \leqq G(3) \leqq 8$; and Hardy is

* Corrected from Hardy's "127" on page 17, in accordance with the information indicated in the next footnote.

† At the time when Hardy wrote his address he was under the impression that Waring first stated his theorem [that every positive integer is a sum of at most 4 positive squares, 9 positive cubes, 19 positive biquadrates, and so on] in the third edition (1782) of his *Meditationes Algebraicae* [pp. 349–350], but in a letter of Jan. 4, 1921, he writes me that a correspondent has called his attention to its appearance in an earlier edition. On examining the three editions I fail to find it in the first (1762), but find it in the second (1770) [pp. 204–205], and in the third (1782), as indicated. These references are given also in Dickson's *History of the Theory of Numbers*, vol. II, pp. xviii and 717 in connection with his elaborate history of Waring's problem. [Am I right in supposing (as I have done in numbering the editions above) that Waring, when he came to publish the third edition, treated the first part (pp. 1–65) of his *Miscellanea Analytica* (1762) as the first edition of his *Meditationes Algebraicae*, its material being reproduced in the editions of 1770 and 1782; or was there another edition of the latter between those of 1770 and 1782? The view which I have taken agrees with a statement given in an old manuscript note on the fly-leaf of a copy of the 1770 edition of the *Meditationes Algebraicae* in the library of the University of Illinois.]

disposed to conjecture that $G(3)$ has the value 4 or 5, and he seems to lean towards the former (p. 23). A simple argument shows that $G(4) \geqq 16$ and Hardy and Littlewood have proved that $G(4) \leqq 21$, so that $G(4)$ lies in a comparatively small known range.

On pages 27–34 we have one of the most fascinating accounts in our literature of the fundamental idea which has guided a mathematical investigation, the account of that which directed Hardy and Littlewood in their investigation of the properties of $G(k)$. The method seems to be very powerful. It has brought them for the first time into relation with the series on which the solution in the last resort depends; it gives numerical results which, as soon as k exceeds 3, are far in advance of any known before; and it gives a definite upper bound to $G(k)$, namely,

$$G(k) \leqq (k-2)2^{k-1} + 5.$$

[On the other side it is known that $G(k) \geqq k+1$ and that $G(2^\alpha) \geqq 2^{\alpha+2}$ if α is a positive integer.] Hardy adds: "It is beyond question that our numbers are still very much too large; and there is no sort of finality about our researches, for which the best that we can claim is that they embody a method which opens the door for more."

Concerning Goldbach's assertion that every even number is the sum of two primes we have the following (p. 34): "Our method is applicable in principle to this problem also. We cannot solve the problem, but we can open the first serious attack upon it, and bring it into relation with the established prime number theory. The most which we can accomplish at present is as follows. We have to assume the truth of the notorious Riemann hypothesis concerning the zeros of the zeta function, and indeed in a generalized and extended form. If we do this we can prove, not Goldbach's theorem indeed, but the next best theorem of the kind, viz. that *every odd number*, at any rate from a certain point onwards, *is the sum of three odd primes*. It is an imperfect and provisional result, but it is the first serious contribution to the solution of the problem."

It is with genuine regret that the reviewer has to point out one or two historical errors in an address which is otherwise so charming. (We have already mentioned one of these.) On page 18 he refers to Fermat's "notorious assertion concerning Mersenne's numbers"; a letter to the reviewer indicates that

this error probably arose through referring to Fermat a statement which was in fact made by Mersenne (and stated by W. W. Rouse Ball to be "probably due to Fermat").*

From page 18 of Hardy's lecture I quote as follows: "No very laborious computations would be necessary to lead Waring to a highly plausible speculation, which is all I take his contribution to the theory to be; and in the theory of numbers it is singularly easy to speculate, though often terribly difficult to prove; and it is only proof that counts." It is hard to see in what sense the author can say that "it is only proof that counts" when he has before him a conjecture like that of Waring which has certainly influenced for good the development of a very fascinating chapter in the modern theory of numbers. Probably the same feeling that induced this statement led to Hardy's calling by the name "theorem of Lagrange" the theorem that every integer is a sum of four non-negative squares, whereas Fermat had stated that he had a proof of the theorem (both Fermat and Bachet ascribing the theorem to Diophantus) and Euler had made repeated efforts for forty years to prove it before Lagrange through the aid of Euler's work succeeded in giving the first proof in 1772. [See Dickson's *History*, vol. II, pp. ix, x, 275–303.] It appears to me to be unfortunate to have this theorem called by the name of Lagrange; it certainly represents one extreme of judgment concerning the question of attaching names of mathematicians to specific theorems.

The opposite extreme of the same thing recently came to my attention in another connection; curiously enough, it is again a case of a "theorem of Lagrange." The theorem that the order of a subgroup is a factor of the order of the group containing it has been called the "theorem of Lagrange" by at least two authors of high repute [see Pascal's *Repertorium* (in German), vol. I, 2d edition, 1910, p. 194, and Miller, Blichfeldt and Dickson's *Finite Groups*, 1916, p. 23 (in the part written by G. A. Miller)]. Now the facts seem to be that

* If any one of the many mathematical propositions stated by Fermat is incorrect, with perhaps a single exception, I am unaware of it. The case of exception is that concerning the prime character of the so-called Fermat numbers $2^k + 1$ where $k = 2^n$; and this incorrect statement he first made several times as a conjecture and finally (after the lapse of several years) implied that he knew a proof of it. This would seem therefore to be a lapse of memory rather than an error in reasoning. All his other theorems have been proved with the one famous exception. See Dickson's *History*, vol. I, p. 375, and vol. II, p. xviii.

Lagrange knew the theorem only for the case of the subgroups of the symmetric group and that even for this case he had no satisfactory proof. Abbati (in 1803) completed the proof for subgroups of the symmetric group and also proved the theorem for cyclic subgroups of any group; but it was apparently more than seventy-five years after the publication of Lagrange's memoir (in 1770–1771) before the completed theorem became current (though it had appeared earlier in a paper by Galois in 1832). In this case we have attributed to Lagrange a theorem which he probably never knew or conjectured, on the ground (it would seem) that he knew a certain special case of it. In Hardy's paper we have a theorem referred to Lagrange apparently on the ground that he first published a proof of it though it had been in the literature long before. Somewhere between these two extremes lies the golden mean of proper practice in attaching the names of mathematicians to specific theorems; and this mean, in the opinion of the reviewer, is rather far removed from each of the extremes indicated.

R. D. CARMICHAEL.

Statics, including Hydrostatics and the Elements of the Theory of Elasticity. By Horace Lamb. Cambridge, University Press, 1916. xii + 341 pp.

Mathematics as ordinarily taught in our colleges and mathematics as used in this work-a-day world are birds of entirely different feather, and they do not flock together. This may perhaps be illustrated by a simple problem (No. 20, p. 178) from Lamb's *Statics*:

"Water is poured into a vessel of any shape. Prove that at the instant when the center of gravity of the vessel and the contained water is lowest it is at the level of the water surface."

Let us imagine a well trained sophomore attacking this problem. It is clearly a minimum problem involving integration. We measure h vertically upward from the bottom of the inside of the container, take the density as unity (or shall we keep it as ρ?), and let $A(h)$ be the area of the cross-section of the vessel. Then the center of gravity of the water is at a height

$$h_1 = \int_0^h \rho h A dh \div \int_0^h \rho A dh.$$

Let the mass of the vessel be denoted by M, and let its center

of gravity be at the altitude h_0. Then the center of gravity of the whole is at the height

$$H(h) = \frac{Mh_0 + \int_0^h \rho h A dh}{M + \int_0^h \rho A dh};$$

and it is to be proved that

$$\frac{dH}{dh} = 0 \quad \text{and} \quad \frac{d^2H}{dh^2} > 0 \quad \text{when} \quad h = H.$$

I have assumed perhaps naturally that the student knows enough or too little not to use double or triple integrals, but knows too much not to use any. I have also assumed that he knows enough to write $h = H$, which I doubt.

Having thus stated his problem, perhaps he can solve it and perhaps he cannot, but probably he will get at least 50% for his statement, even if he has not proved anything except that his thorough course in calculus has developed a serious set of inhibitions whereby he is prevented from using his common sense on a simple mathematical problem.

Lamb's Statics is full of such disappointments for the student of mathematics. Indeed it may be feared that this is malice aforethought on the part of the author. (See ¶ 7 of his Preface). Nor would I seem to criticize mathematical instruction of others without being more precisely critical of my own. For twenty years I have taught or tried to teach not only mathematics of the canonical sort to all grades of students but mechanics and physics to Freshmen, to Juniors, to Seniors, and to Graduates. It is my own experience that despite my best endeavors my students will solve successfully with complicated mathematical machinery problems that I am confident they do not understand and will fail lamentably in the solution of simple common-sense problems where the canonical machinery is in the way. Perhaps we should teach less of the machinery, reduce things less to rule and formal procedure, and above all dwell longer on the simple fundamentals of our subjects of instruction. Mechanics is a particularly difficult topic for teacher and for taught. In our large engineering schools we are apt to take three shots at mechanics: once in the course in physics, once at some point of the course in mathematics, and once in applied mechanics. The material covered overlaps a great deal, but the points of view are often divergent. Might it not be that if we all got together, physicist, mathematician, and engineer, pooling our

combined time, and cooperating at each phase of the work, we should accomplish a far better result?

Lamb's Statics keeps to the middle road of presenting the subject as a branch of natural science, largely deductive because of the paucity and simplicity of the fundamental laws. It is a work on physics rather than on engineering or mathematics; it should afford a fine introduction whether to applied mechanics of the more technical sort, to the theory of structures, to more advanced physical theories, or to analytic statics.

To show the breadth of treatment the titles of the chapters may be quoted: Theory of Vectors, Statics of a Particle, Plane Kinematics of a Rigid Body, Plane Statics, Graphical Statics, Theory of Frames, Work and Energy, Analytical Statics, Theory of Mass-Systems (centers of mass and moments of inertia), Flexible Chains, Laws of Fluid Pressure, Equilibrium of Floating Bodies, General Conditions of Equilibrium of a Fluid, Equilibrium of Gaseous Fluids, Capillarity, Strains and Stresses, Extension of Bars, Flexure and Torsion of Bars, Stresses in Cylindrical and Spherical Shells. This is a considerable program; it is well and consistently carried through —as should be expected by all who have known his other writings and particularly his companion volume on *Dynamics*, noteworthy for the same Greek characteristic *σωφροσύνη*.

These books, *Statics* and *Dynamics*, are not written for the writing; they are products of teaching, for they are based on lectures delivered at the University of Manchester. If the matter were taken slowly enough, satisfactory results would attend their use in our American institutions, provided our teachers had an all round interest in the elements of mathematics, of physics, and of engineering, and a fine contempt for superficial ground-covering in any of the three.

E. B. Wilson.

Principes usuels de Nomographie avec applications à divers problèmes concernant l'artillerie et l'aviation. Par Lieutenant-Colonel Maurice d'Ocagne. Paris, Gauthier-Villars, 1920. 67 pages.

This pamphlet, as the title indicates, is a short exposition of nomography in which the illustrations are taken from artillery and aviation. Nomography is the general theory of the graphical representation of equations of any number of vari-

ables: the end of such a representation is to replace all kinds of numerical computation by readings made on the graph. If an equation contains three variables x, y, z, say, then by giving a value to z and using x and y as rectangular coordinates a curve can be constructed. If these curves are constructed for various values of z then a set of values x, y, z satisfying the equation will be found by taking the coordinates x, y of any point together with the value of z which corresponds to the curve passing through that point. If the z-curves are constructed reasonably close together it is possible to interpolate. The corresponding values are then determined as the intersection of three curves. If then instead of point coordinates, tangential coordinates are used corresponding values will correspond to points on a line. D'Ocagne has introduced a particular kind of tangential coordinates which makes the constructions very simple. This is the alignment chart and has many advantages over the construction first mentioned. The drawing of curves is replaced by the construction of scales and, for the case of four variables, the whole drawing can be made on a single sheet of paper. The first half of the pamphlet deals with the theory of the construction of these charts.

The second part applies the theory to the actual making of charts for formulas used in artillery and aeronautics. The charts are arranged according to the simplicity of the construction. Among the charts we find the following: the coefficient of adjustment, $K = D/A$; the angle of sight, $\tan z = Z/D$; the initial angle of fire,

$$(1 + \cos 2\varphi) \tan \epsilon = \sin 2\varphi - \sin 2\alpha;$$

the limiting velocity for arming fuses, $\frac{1}{4}K\pi D^2 = p/V^2$; the interior pressure in the autofrettage,

$$\omega + 1492 = \frac{\lambda_1}{\rho^n} + \frac{\lambda_2}{\rho^2}.$$

These give an idea of the kind of formulas handled. The charts are well constructed and the explanations are very clear. On the whole this little pamphlet makes delightful reading and, now that graphical charts are so much used, the teacher of mathematics should no longer be ignorant of the subject.

C. L. E. MOORE.

Mathematische Streifzüge durch die Geschichte der Astronomie. By P. Kirchberger. Mathematische-Physikalische Bibliothek, Bd. 40. Leipzig, Teubner, 1921. iv + 54 pp.

Some of the difficulties of publication and sale in connection with German works are seen in the case of this worthy addition to the Teubner list,—one of the small monographs of the type that was intended to and did popularize science before the war, and that sold in those days for a mark or less, often in a cloth binding. This particular number, like others at the present time, appears in paper binding, with a nominal price of 2 Marks in Leipzig, but with the announcement that there is also a Teuerungszuschlag of 120% for home purposes, with a Bewilligung stamp saying that the value is 3 Marks 30 Pf., and with an accompanying statement that the price of the book to an American buyer is 10 Marks,—all of which is an interesting problem for the economist.

As to the work itself, it maintains the high intellectual standard usually found in German booklets of this class. It is written by one who holds professorial rank in the Leibniz-Oberrealschule at Charlottenburg, and it shows that scholarly ideals still prevail in spite of post-bellum difficulties.

Dr. Kirchberger has divided his work into six chapters, in each of which he has considered the essential features which mathematics has contributed to the progress of astronomy. The general topics are as follows.

I. Early history to the time of Hipparchus, including (a) the first steps in the making of a system of chronology, together with early traces of the idea of a continued fraction; (b) astronomical geography, as in the geodetic work of Eratosthenes; (c) the Greek contributions to measurements relating to the sun and the moon, as in the finding of the supposed size of and distance to the sun by Aristarchus.

II. The planetary theories of Ptolemy, Copernicus, and Tycho Brahe, including Ptolemy's theory of epicycles and the effect of improved methods of observation and computation upon the work of the sixteenth century.

III. Kepler, with a résumé of his leading mathematical contributions to astronomy, and with attention to the fact that he looked upon the latter science as a branch of physics rather than a part of geometry.

IV. Newton, with especial reference to his theory of gravity.

V. From Halley to Bessel, including particularly the de-

velopment of the modern ideas of parallax and aberration and with special attention to the contributions of Lacaille, Lalande, and Römer.

VI. Modern problems.

The work is, of course, merely the briefest kind of survey. Only forty-one names are mentioned, and under the prominent names there could not be, under the limitations imposed, more than a mere indication of the work accomplished by each. Nevertheless, for general students of mathematics and of astronomy, the book will furnish some interesting reading and some valuable information. There is a helpful bibliography, mostly of German sources, but including several translations from the English and French.

DAVID EUGENE SMITH.

Natural Tangents. By Emma Gifford. Manchester, England, 1920. iv + 90 pp.

It is now seven years since Mrs. Gifford issued her table of natural sines to every second of arc and to eight places of decimals. This appeared in the spring of 1914, before the war began. The demands upon the time and strength of everyone in England were so great during the years that followed that Mrs. Gifford's projected table of natural tangents was greatly delayed, and the present volume is only the beginning of a work that will be as complete as was the former one. The table extends only to 15°, but the progress will now be more rapid and we may hope soon to see the work completed.

Mrs. Gifford has taken the natural tangents for every 10″ from the *Opus Palatinum* of Rheticus, published in 1596. She has then found the tangents to 1″ through interpolation by the aid of a calculating machine. The computation was in each case carried to ten places for the purpose of establishing the eighth place. The arrangement is that of the Chambers logarithmic tables, which is more convenient than the semi-quadrantal plan commonly found in this country, although it makes the book more bulky.

Mrs. Gifford is so well known as a careful computer that the work will be welcome to all who have need for an eight-place table of natural tangents. Such a table is suggestive of the diminishing relative importance of the logarithm with the rapid improvement in mechanical calculation.

DAVID EUGENE SMITH.

Oeuvres Complètes de Cristiaan Huygens, publiées par la Société Hollandaise des Sciences. Tome XIV. *Calcul des Probabilités. Travaux de Mathématiques Pures, 1655–1666.* La Haye, Martinus Nijhoff, 1920. 524 pp.

Approximately the first one-third (pp. 1–179) of this attractive volume is devoted to contributions to the theory of probability (1656–1657), with appendices most of which are of later date. The next division (pp. 181–407) presents contributions to various subjects (1655–1659) including the theory of numbers, rectification of the parabola, quadrature of conoids, volumes of solids of revolution, centers of gravity, properties of the cycloid, and theory of evolutes. The third division (pp. 409–427) gives contributions to the commentaries of von Schooten on the *Geometria* of Descartes, editions of 1649 and 1659. The fourth division (pp. 429–524) like the second relates to a wide range of subjects. This division (1661–1666) presents contributions to the theory of logarithms including the logarithmic curve with application to the determination of altitude by means of the barometer, further work on solids of revolution, rules for finding the tangent to an algebraic curve, and the theory of cubics. Each division is preceded by a valuable historical introduction.

Although the first steps in the theory of probability were taken by Pascal and Fermat in their correspondence as early as 1654 about questions proposed by the Chevalier de Méré, the work of Huygens constitutes the first treatise on the subject. He starts from the fundamental assumption that there exists for any equitable game a determinate numerical value for the probability that a player will win or lose. He develops a few elementary propositions of which he makes frequent application. Proposition III asserts that if a player has "p chances of gaining a and q chances of gaining b, his expectation is $(pa + qb)/(p + q)$." In present day usage, we should probably avoid the use of the word "chances" in the sense in which it is here used. The thought could be expressed by saying that $(pa + qb)/(p + q)$ is the expectation of a trial when in a total of $p + q$ equally likely trials, there are p ways of gaining an amount a and q ways of gaining b. This proposition affects much that follows it.

The entire treatment of probability contains fourteen propositions, five problems, and nine appendices. The first two propositions are special cases of proposition III just

stated. The propositions IV to IX are concerned with the problem of points with two or three players. The propositions X to XIV deal with the throwing of dice. The nine appendices treat various applications many of which came up in correspondence. The last one, which carries the date 1688, is the problem of the chances of three players at piquet. This problem was reduced by Huygens to the sum of an infinite series. This appendix was written much later than the treatise.

Apart from the division on probability, the volume is devoted mainly to geometry. However, there are about seventeen pages of text given to number theory. This work on number theory relates mainly to the solution of the equation of Pell, and closely related problems. The contributions to geometry vary in character from material as elementary as a new demonstration of the Pythagorean proposition to the determination of surfaces of parabolic and hyperbolic conoids, the tangents to algebraic curves, and the theory of evolutes. Considerable space is given to properties of the cycloid. In this work, Huygens was probably accepting the challenge put up by the letter of Pascal (under an assumed name) addressed "to all geometers of the universe" in which he proposed problems about a certain semisegment of the cycloid calling for the area, the center of gravity, volume of solids generated by revolution about certain lines, and the centers of gravity of these solids. Among the geometric contributions, we should perhaps mention especially that this volume gives properties of the logarithmic curve, and the quadrature of the hyperbola by the use of logarithms.

The division on probability is given both in Dutch and in a French translation. The text of the other divisions is, with a few exceptions, in Latin. The valuable footnotes and introductions to divisions are in French. The volume contains a table of contents, a name index, and a subject index that are very useful. When we consider the preparation of the introductions to the various divisions of the work, the large number of references, and the many footnote explanations and comments found throughout the work, it seems to the reviewer that this volume represents practically a model piece of work from the editorial standpoint as well as a contribution to the history of Huygens's mathematical work.

H. L. Rietz.

Dynamics of the Airplane. By K. P. Williams. New York, John Wiley and Sons. 136 pages.

This little volume is number 21 of the series of Mathematical Monographs edited by Mansfield Merriman and R. S. Woodward. The author states that it is an outgrowth of a set of lectures delivered at the University of Paris by Professor Marchis for the benefit of the students of the American Army.

The subject of aeronautics as usually given includes both rigid mechanics and fluid mechanics. The latter, however, is not of so much importance for elementary considerations and has been omitted from the book under consideration. The omission does not interfere with the understanding of the dynamics of the airplane even when we include Bryan's mathematical theory of stability and Wilson's treatment of the effect of gusts. Almost all mathematical problems of the airplane can be treated with no more fluid mechanics than is usually given in a good college course in physics.

The more elementary subjects usually considered in a book of this character are: pressure on a plane and curved surface, movement of the center of pressure as the inclination of the plane is varied, the relation of the velocity of flight to the angle of incidence of the wing, power consumed, ascent, descent and circular flight. These subjects are adequately and delightfully treated in the first four chapters, 68 pages. The fifth chapter is devoted to the propeller. The principle topics discussed are form of the blade, thrust, power and efficiency. The formulas obtained for these are

$$T = \alpha V^2 D^2, \qquad P = \beta n^2 D^5, \qquad E = TV/(550\,P),$$

where n denotes r.p.s., D denotes the diameter of the propeller, α and β are functions of V/nD. The last part of the chapter is devoted to the experimental determination of the functions α and β and the adaptation of the propeller to the machine. Chapter VI, on performance, discusses the highest attainable altitude and the radius of action together with the effect of loading on these. Chapter VII on stability and controlability discusses in a general way the meaning of stability and the effect of stabilizer, elevator, etc. Chapter VIII treats Bryan's theory of stability. The rigid mechanics necessary is not developed here but reference is made to Routh.

The book is well written, very attractive in appearance, and makes a splendid introduction to the subject.

C. L. E. Moore.

The Sumario Compendioso of Brother Juan Diez: The Earliest Mathematical Work of the New World. By David Eugene Smith. Boston, Ginn and Company, 1921. vii + 65 pp.

To the historian and to the bibliophile the publication of this little volume will be an event of importance. But even to one who, like the reviewer, can lay no claim to such distinction the possession and perusal of the work will be a source of genuine interest and pleasure.

The original was printed in the City of Mexico in 1556 "long before the migration of the pilgrims to this continent was thought of, a half century before Henry Hudson discovered the river that bears his name, and nearly two centuries before the first mathematical work in English appeared from the pen of an American scholar." The author was one Juan Diez, a priest and companion of Cortez in the conquest of Mexico.

The greater part of the original consists of an elaborate set of tables intended to minimize as far as possible the computations arising in the sale of gold and silver. Only one page of these tables is reproduced in the present edition. All of the remainder of the work, however, consisting of a brief textbook on arithmetic "suited to the needs of apprentices in the counting houses of the New World" and of six pages of algebra (devoted largely to the quadratic equation), is reproduced in facsimile with an English translation appearing on the opposite pages. When it is recalled that only a small number of books on algebra had appeared in Europe before 1556 it is indeed "remarkable"—as the editor points out—"that an obscure writer in Mexico should have produced even six pages on the subject in this early period in the development of printed scientific literature."

The fact that only four copies of the original are known to exist made it desirable to make a facsimile reproduction of the work more generally available. This has been made possible through the cooperation of the distinguished editor and translator, an anonymous "public spirited gentleman" who subscribed to the greater part of the limited edition in advance of publication, and of a publishing house which has in the past given evidence not merely of high proficiency in the art of book-making but also of scholarly ideals not unduly influenced by the balance-sheet. To them the cordial thanks of the scientific fraternity are due.

J. W. YOUNG.

Theoretische Arithmetik. Von Otto Stolz und J. A. Gmeiner. Zweite Auflage, bearbeitet von J. Anton Gmeiner. I. Abteilung, 1911, vi + 146 pp.; II. Abteilung, 1915, viii + 369 pp. Leipzig, Teubner.

Maintaining the same division into "Abschnitte" as in the first edition and largely the same separation of each into sections, Gmeiner has given us in the second edition of the *Theoretische Arithmetik* of Stolz and Gmeiner a work larger by about twenty-five percent than the first edition of one volume. In every Abschnitt one finds numerous modifications and extensions, often of a minor character but occasionally of considerable extent. Many of the modifications are in the direction indicated by Stolz in notes he had made looking forward to a revision of the text. Gmeiner has carefully utilized these suggestions so as to carry out as far as possible the wishes of his teacher in this new edition of their common work.

The larger changes may be indicated briefly as follows: There is a fuller treatment in the second Abschnitt of subtraction of integers, division, powers, and the systematic representation of numbers; and there is a new section here on zero as a number. In the third Abschnitt the theory of the laws of operation has been expounded anew and in fuller form, especially that having to do with the distributive law. The sections on the fundamental operations with real numbers in the seventh Abschnitt and the theory of complex units in the tenth Abschnitt appear in a new form. Other changes, less important, are to be found throughout the volumes. In its new form the book will have an increased usefulness.

As the work is now divided the first volume treats the theory of rational numbers and the second the theory of real and complex numbers together with an introduction to the theory of infinite series both with real terms and with complex terms.

R. D. CARMICHAEL.

Advanced Lecture Notes on Light. By J. R. Eccles. Cambridge, University Press, 1919. 141 pp.

A sequel to the author's *Lecture Notes on Light*, 1917, the *Advanced Lecture Notes* treat in sequence Rainbows, Magnifying Power, Chromatic Aberration, Spherical Aberration, Wave Theory of Light, Interference, Diffraction, Polarisation of Light. There are really only half the indicated number of pages, because the left-hand page is left blank apparently

with the intention that the student shall draw thereon the figures which are entirely lacking in the text and perhaps make certain calculations supplementary to or illustrative of the text. A perusal of the work fails to indicate surely whether the lectures as given are largely theoretical or are amply illustrated by experiment showing the phenomena on a scale impressive even to undergraduates. No part of physics lends itself so well to treatment by experimental lectures as optics, particularly physical as contrasted with geometrical optics.

It is unnecessary to commend either the excellent care given to the treatment or the selection and order of the topics. The discussion of geometrical optics can hardly be made so moving as that of physical optics, but every pedagogic instinct, except that of fascinating or even mystifying the student, is better satisfied by commencing with the former, as the author does, and so far as the real training of the student in analysis or in the things of most use to him is concerned, geometrical optics will long remain preferable to physical. It would have been desirable, provided the students' previous training sufficed, to lay some small emphasis on the dynamics of wave motion.

The question of what shall be taught in optics and how it shall be taught is like the corresponding questions relative to mechanics, electricity, and heat, not only unsolved but as yet unstated in a form capable of solution. Eccles' book merits careful consideration by collegiate teachers of physics.

E. B. WILSON.

Materialien für eine wissenschaftliche Biographie von Gauss. Gesammelt von F. Klein, M. Brendel, und L. Schlesinger Heft VIII. *Zahlbegriff und Algebra bei Gauss.* Von A. Fraenkel. Mit einem Anhang von A. Ostrowski: *Zum ersten und vierten Gaussschen Beweise des Fundamentalsatzes der Algebra.* Erster Teil. Leipzig, Teubner, 1920. 58 pp.

The title of this pamphlet is sufficiently explanatory of its general character. As it forms only the first part of the eighth volume of the series, it is incomplete in some respects. However, it does contain an interesting and well rounded discussion of Gauss's part in the development of the concept of number and an arithmetization of his first proof of the fundamental theorem of algebra. A corresponding treatment of the fourth proof, together with detailed criticisms of the two proofs, are apparently to be given in the second part of the volume.

As the author remarks, there can be no doubt that Gauss recognized the sphere of philosophy in investigating the foundations of mathematics, and it is interesting to note that he opposed Kant's view that space is merely a creation of our senses. While Gauss's part in the development of the complex number was probably of relatively more importance, it is pointed out that he made substantial contributions to the modern theory of real numbers. The somewhat controversial subject of the priority of different writers with regard to the former seems on the whole to be treated impartially, although Argand is given less credit than some would doubtless accord him.

It is worth noting that Gauss himself recognized the somewhat unsatisfactory character of his first proof of the existence of roots of an algebraic equation. Any comments on Ostrowski's discussion would seem more appropriate in connection with Fraenkel's criticism of the original proof.

HOWARD H. MITCHELL.

ERRATA.

Vol. XXVI, p. 292, formula (22): Instead of $D=2(n-1)\cdots$ read $D = -2(n-1)\cdots$.

——, p. 293, formula (25): The denominator of the last integral in the value for x should be s^2; and, in the value for y, s^3.

Vol. XXVII, p. 11, line 6 of § 1: Instead of the words *of all functions* read *of all bounded functions*.

——, p. 11, formula (2): Add after the formula $(a < y < b)$.

——, p. 17, line 17: Add after the last sentence the sentence: In order to make sure that the integral (1) shall belong to the class $[f]$, it is necessary to assume also that $K(x, y) - K(x, a)$, considered as a function of x, belongs to that class for every value of y.

——, p. 326: Professor R. L. Borger desires to withdraw, at least tentatively, the theorem he announced on this page, on account of a flaw in the proof near the bottom of page 327, which was called to his attention by Dr. T. H. Gronwall.

——, p. 364, line 17: Instead of $m = -9$ read $m = -91$.

——, p. 385, last line: Instead of Granier read Garnier.

NOTES.

At the request of the Council of the Society, the Committee of Publication has decided to begin the next volume of this BULLETIN (vol. XXVIII) with the number to be dated January, 1921. Hence the usual October, November, December numbers will not be issued this fall. Volume XXVIII will consist of ten numbers as usual, however, with mere change of dates. This change of dates has been contemplated for several years; — it will be convenient because the BULLETIN year will hereafter coincide with the calendar year, with the fiscal year of the Society, and with the terms of office of the editors and other officers. This year the change can be made without any real delay, for the printers' strike has so far deferred publication that to date the next number in January will only correspond to the actual fact. Subscribers to the new volume should notice that they will receive the usual number of issues, and that they will actually receive them quite as soon as would be possible otherwise.

The annual meeting of the Society will be held in Toronto on December 28–29, in connection with the sessions of the American Association for the Advancement of Science. The usual Christmas meeting at Chicago, as well as the annual meeting usually held in New York, has been transferred to Toronto, and will be merged with the annual meeting. Titles and abstracts of papers for this meeting should be sent to the Secretary of the Society, R. G. D. Richardson, Brown University, and must be in his hands not later than December 10, 1921.

At one of the sessions of the Toronto meeting, Professor R. D. Carmichael will give an address as retiring chairman of the Chicago Section. The subject will be *Algebraic guides to transcendental problems.*

The fourteenth regular meeting of the Southwestern Section of the Society will be held in St. Louis at Washington University on Saturday, November 25. Titles and abstracts of papers for this meeting must be in the hands of the Secretary of the Section, E. B. Stouffer, University of Kansas, not later than November 5, 1921.

A list of dates of meetings and other announcements is printed on the inside of the back cover.

The April number (vol. 22, no. 2) of the TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY contains the following papers: *On division algebras*, by J. H. M. Wedderburn; *Oscillation theorems for the real, self-adjoint linear system of the second order*, by H. J. Ettlinger; *New proofs of certain finiteness theorems in the theory of modular covariants*, by Olive C. Hazlett; *On the convergence of certain trigonometric and polynomial approximations*, by Dunham Jackson; *Determination of all general homogeneous polynomials expressible as determinants with linear elements*, by L. E. Dickson; *Pseudo-canonical forms and invariants of systems of partial differential equations*, by A. L. Nelson; *Arithmetical paraphrases* (II), by E. T. Bell; *On the zeros of solutions of homogeneous linear differential equations*, by C. N. Reynolds, Jr.; *A generalization of the Fourier cosine series*, by J. L. Walsh; *Polynomials and their residue systems*, by A. J. Kempner.

The April number (vol. 43, no. 2) of the AMERICAN JOURNAL OF MATHEMATICS contains: *Boundary value and expansion problems: algebraic basis of the theory*, by R. D. Carmichael; *Algebraic theory of the expressibility of cubic forms as determinants, with application to diophantine analysis*, by L. E. Dickson; *The impossibility of Einstein fields immersed in flat space of five dimensions*, by Edward Kasner; *Finite representation of the solar gravitational field in flat space of six dimensions*, by Edward Kasner; *On the motion of two spheroids in an infinite liquid along their common axis of revolution*, by Bibhutibhusan Datta.

The mathematical seminar of the University of Warsaw has founded a new periodical, FUNDAMENTA MATHEMATICÆ, devoted entirely to the theory of assemblages and related subjects, such as analysis situs, mathematical logic and theory of axioms. Articles offered for publication should be sent to the editors, S. Mazurkiewicz and W. Sierpiński, at the University of Warsaw, and must be written in French, Italian, English, or German. Two volumes have already appeared.

The Society of German Engineers announces that it will publish, beginning early in 1921, a new periodical devoted to applied mathematics, the ZEITSCHRIFT FÜR ANGEWANDTE MATHEMATIK UND MECHANIK. The editors are R. von Mises (editor in chief), A. Föppl, R. Mollier, H. Müller-Breslau, L. Prandtl, and R. Rüdenberg.

Attention is called to the announcement, on one of the fly-leaves of this number of the BULLETIN, concerning the JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES, often called LIOUVILLE'S JOURNAL or JORDAN'S JOURNAL. The success of the effort to maintain this important journal is of interest to mathematicians everywhere. Gratifying response has been received already by friends of the JOURNAL in this country.

The summer meeting of the British Association for the Advancement of Science will be held at Edinburgh, September 7–14. Professor O. W. Richardson has been appointed president of Section A (mathematics and physics) for this meeting.

A meeting of the National Academy of Sciences was held in Washington, April 25, 26 and 27, 1921. Professor Albert Einstein and the Prince of Monaco were guests of honor at the annual dinner. Professor G. A. Miller and Professor A. E. Kennelly were elected to membership. The following mathematical papers were read: by Edward Kasner, *A model of the solar gravitational field;* by G. D. Birkhoff, *On the problem of three or more bodies;* by L. E. Dickson, *Quaternions and their generalizations;* by L. E. Dickson, *Investigations in algebra and number theory;* by H. F. Blichfeldt, *On the approximate solution in integers of a set of linear equations.*

At a meeting of the American Philosophical Society in Philadelphia, April 21, 22 and 23, Professor G. D. Birkhoff was elected to membership. The following papers were read by members of the American Mathematical Society: by L. E. Dickson, *The atomic theory and ideal numbers;* by J. A. Miller, *Application of the interferometer to the determination of stellar parallaxes;* by F. R. Moulton, *Recent astronomical explorations in space and in time;* by M. B. Snyder, *1. Universal volcanism and the cosmic atomic numbers; 2. Planck's constant "h," a variable.*

As a result of the preliminary conference held in January (BULLETIN, p. 336) a committee of the National Research Council on the mathematical analysis of statistics has been organized, consisting of H. L. Rietz (chairman), J. W. Glover, E. V. Huntington, T. L. Kelley, R. Pearl, and W. M. Persons.

The lectures on the theory of relativity delivered by Professor Albert Einstein at Princeton (BULLETIN, p. 386) will be published by the Princeton University Press. Professor Einstein lectured at the University of Chicago on May 3, 4 and 5.

Professor Emile Borel has been elected a member of the Paris Academy of Sciences in the section of geometry, as successor to the late Professor Georges Humbert. Sir George Greenhill has been elected correspondent in the section of mechanics, as successor to the late Professor W. Voigt.

Professor Albert Einstein has been elected a foreign member of the Royal Society of London.

The honorary degree of doctor of science has been conferred by the University of Manchester on Dr. Horace Lamb and Sir Ernest Rutherford.

The Royal Society of Edinburgh has awarded the Macdougall-Brisbane prize for the period 1918–1920 to Professor J. H. M. Wedderburn of Princeton, in recognition of his investigations on hypercomplex numbers.

Professor E. Naetsch has been appointed professor of analytic geometry at the Technical School at Dresden, not at the Technical School at Darmstadt, as was announced in the May number of this BULLETIN.

Dr. G. Bouligand, of the University of Rennes, has been appointed professor of rational mechanics at the University of Poitiers.

Sir Asutosh Mookerjee (Mukhopâdhyáy), president of the Calcutta Mathematical Society, has been appointed vice-chancellor of Calcutta University.

At a general meeting of the members of the Royal Institution held May 9, Sir J. J. Thomson was elected honorary professor of natural philosophy, and Sir Ernest Rutherford professor of natural philosophy.

Mr. T. A. Brown has been appointed senior lecturer in mathematics for 1921–1922 at University College, London.

It is reported that Professor H. Lamb is to be appointed to an honorary lectureship to be called the Rayleigh lectureship in mathematics, at Cambridge University.

Arrangements for exchange professorships in engineering and applied sciences have recently been completed between certain French and American universities and technical schools. Professor A. E. Kennelly, of Harvard University and the Massachusetts Institute of Technology, has been chosen as the first American representative.

One of the recently endowed Sterling professorships at Yale University has been assigned to Professor E. W. Brown.

Assistant Professor P. J. Daniell, of the Rice Institute, has been promoted to a full professorship of applied mathematics.

Assistant Professor H. M. Robert, Jr., of the United States Naval Academy, has been promoted to an associate professorship of mathematics.

At the University of Alberta, Associate Professor S. D. Killam has been promoted to a full professorship and Assistant Professor J. W. Campbell to an associate professorship.

At Cornell University, Professor James McMahon retires at the close of the current academic year. The previously announced Heckscher Research grant to Mr. H. S. Vandiver for the continuation of his investigations on the theory of algebraic numbers has been extended to cover the first term of the year 1921–22, and Mr. Vandiver has been granted leave of absence for that period. Mr. Jesse Osborn of Pennsylvania State College and Mr. R. L. Jeffery of Acadia College have been appointed instructors.

At Clark University, Professors W. E. Story, Henry Taber and Joseph de Perott retired in June.

At Dartmouth College, Professor C. N. Haskins has been appointed chairman of the department of mathematics for a term of two years. Assistant Professor R. D. Beetle has been promoted to a full professorship. Professor E. G. Bill has been appointed Dean of Freshmen; he retains his professorship, but will not give instruction during the academic year 1921–22. Professor J. W. Young returns to the department after two years' leave of absence during which he has served as chairman of the National Committee on Mathematical Requirements. Assistant Professor F. M. Morgan has been granted leave of absence for the second semester of the academic year 1921–22.

Dr. J. F. Ritt, of Columbia University, has been promoted to an assistant professorship of mathematics.

Mr. H. L. Smith, of the University of Wisconsin, has been appointed professor of mathematics at the University of the Philippines.

Dean P. C. Porter, of the Baylor College has been appointed assistant professor of mathematics at Baylor University.

Associate Professor E. D. Grant, of the Michigan College of Mines, has been appointed head of the department of mathematics at Earlham College.

Mr. J. A. Herrington, of the University of Wisconsin, has been appointed associate professor of mathematics and engineering at Beloit College.

Mr. T. L. Hamlin, of Union College, has been appointed head of the department of mathematics at the Clarkson College of Technology, Potsdam, N. Y., as successor to Professor H. M. Royal, resigned.

At the College of Wooster, Mr. C. O. Williamson has been promoted to an assistant professorship of applied mathematics.

Assistant Professor F. N. Bryant, of the State College of Washington, has been appointed assistant professor of mathematics at Syracuse University.

At the College of William and Mary, Mr. J. S. Counselman has been appointed professor of mathematics and Mr. J. C. Lyons assistant professor.

Professor J. B. Hamilton has been appointed head of the department of mathematics at the University of Tennessee.

Mr. J. P. Beaman has been appointed head of the department of mathematics at the Kentucky Wesleyan College, as successor to Professor C. S. Venable, resigned.

Mr. F. J. Rogers has been appointed head of the department of mathematics at Shaw University, Raleigh, N. C.

Professor L. S. Dancey has been appointed head of the department of mathematics at Carroll College, Waukesha, Wis., as successor to Professor S. B. Ray, resigned.

Mr. L. C. Bagley, of Washburn College, has been appointed professor of mathematics at Ottawa University.

Mr. Thurman Andrew has been promoted to a professorship at the West Virginia Wesleyan College, as successor to Professor C. E. White, resigned.

In October, 1920, Mr. F. A. Lewis, of the Texas Agricultural and Mechanical College, and Dr. C. G. P. Kuschke were appointed assistant professors of mathematics.

At The Citadel, the military college of South Carolina, Major R. G. Thomas, professor of mathematics and engineering for thirty-one years, has been made professor emeritus. He was succeeded by Colonel O. J. Bond.

At Fairmount College, Wichita, Mr. H. G. Titt, of Yankton College, has been appointed professor of mathematics and dean of the College of Liberal Arts.

At Villanova College, Reverend J. S. O'Leary has been appointed professor of mathematics, as successor to Reverend F. A. Driscoll, who has been made president of the college.

Mr. E. K. Paxton has been appointed assistant professor of mathematics at Washington and Lee University.

Professor A. H. Norton, of Elmira College, has been appointed president and head of the department of mathematics at the Keuka College for Women.

Professor E. F. Canaday, of the University of South Dakota, has been appointed professor of mathematics at Meredith College.

At Central Wesleyan College, Warrenton, Mo., Professor J. H. Frick, head of the department of mathematics for fifty years, has retired. He was succeeded by Mr. H. V. Knorr.

Dr. V. H. Wells has been appointed assistant professor of mathematics at Carleton College.

At the University of Oklahoma, Dr. J. O. Hassler, of the Crane Junior College, has been appointed associate professor of mathematics. Assistant Professor H. C. Gossard has resigned, to enter Y. M. C. A. work Assistant Professor E. D. Meacham has been granted leave of absence for 1921–22.

Mr. F. H. Murray, of Harvard University, has been awarded one of the American Field Service fellowships for study in the French universities.

Recent appointments as instructor in mathematics in American colleges and universities are as follows: Colgate University, Mr. Torvald Frederiksen; Columbia University, Dr. E. L. Post; Cornell College, Mt. Vernon, Iowa, Mr. G. F. Rouse; Denison University, Miss Grace E. Jefferson; Illinois College, Miss Lois Daniels; Lafayette College, Mr. R. J. W. Templin; Princeton University, Mr. Philip Franklin; Randolph-Macon College, Mr. B. F. Walton; Randolph-Macon Woman's College, Miss Virginia Watts; Ripon College, Mr. William Bollenbeck; Shorter College, Miss Jennie D. Ehlers; State College of Washington, Miss Gladys H. Freeman and Mr. H. H. Irwin; Tufts College, Miss Edith L. Bush and Mr. B. A. Hazeltine; University of Maine, Mr. A. I. Bless; University of Nebraska, Mr. J. H. Taylor; University

of Oklahoma, Mr. H. G. Lieber, Miss Dora McFarland and Miss Ella Mansfield; University of Pittsburgh, Mr. F. J. Burkett; University of Vermont, Mr. F. W. Householder and Mr. H. G. Millington.

From an account of the life and work of Staeckel sent by Professor W. Lorey, the following extracts are selected:* Paul Staeckel was born at Berlin, August 20, 1862, and died December 12, 1919. He studied at the University of Berlin, having among his teachers Kronecker, Weierstrass, Helmholtz and Fuchs; he received his doctorate in 1884 and his teacher's certificate in 1885. He taught at the Berlin Wilhelm gymnasium and at the universities and technical schools of Halle, Königsberg, Hanover, Karlsruhe and Heidelberg. Among his scientific researches are studies on mechanics, arithmetic properties of analytic functions and the representation of even numbers as sums and differences of primes. He edited the fourth volume of the German report of the International Commission on the Teaching of Mathematics, translated Borel's *Eléments de Mathématiques*, and exhibited elsewhere much interest in pedagogy. His contributions to the history of mathematics include editorial service in connection with Euler's collected works and with Ostwald's mathematical classics; a collection of original sources on the theory of parallels, in collaboration with Engel; a study of the geometric work of Gauss; and numerous articles in Bibliotheca Mathematica.

Professor A. M. G. Floquet, of the University of Nancy, died October 7, 1920, at the age of seventy-two years.

Colonel Vallier, correspondent of the Paris Academy of Sciences in the section of mechanics and well known for his work in ballistics, died March 29, 1921.

Professor Alfred Doolittle, of the Catholic University, died February 23, 1921.

Mr. J. S. Hoffman, instructor in mathematics at Tufts College, died June 29, 1920, at the age of twenty-three years.

* A full account by Professor Lorey appears in the ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT ALLER SCHULGATTUNGEN, vol. 52, pp. 85–88.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BERZOLARI (L.). Geometria analitica. I: Il metodo delle coordinate. 2a edizione. (Manuali Hoepli.) Milano, Hoepli, 1920. 16mo. 14 + 496 pp.

BHATTACHARYYA (D.). Vector calculus. (Griffith Prize thesis.) Calcutta, University Press, 1920. 8vo. 90 pp.

BIBLIOTHECA chemico-mathematica: catalogue of works in many tongues on exact and applied science, with a subject index. Compiled and annotated by H. Zeitlinger and H. C. S. London, Sotheran, 1921. 2 volumes. 8vo. 12 + 964 pp. £ 3 3s.

CARNOT (L.). Réflexions sur la métaphysique du calcul infinitésimal. 2 volumes. (Les Maîtres de la Pensée Scientifique.) Paris, Gauthier-Villars, 1921. 16mo. 8 + 117 + 105 pp. Fr. 3.50 + 3.50

CLAIRAUT (A. C.). Eléments de géométrie. 2 volumes. (Les Maîtres de la Pensée Scientifique.) Paris, Gauthier-Villars, 1920. 16mo. 14 + 95 + 103 pp. Fr. 3.50 + 3.50

DIEZ (J.). See SMITH (D. E.).

ENGEL (F.). See EULER (L.).

EULER (L.). Opera omnia. Series I: Opera mathematica. Leipzig, Teubner. II–III: Commentationes arithmeticæ. Edidit F. Rudio. Volumen 1, 38 + 611 pp., 1915. Volumen 2, 38 + 543 pp., 1917. XIII: Institutiones calculi integralis. Ediderunt F. Engel et L. Schlesinger. Volumen 3, 18 + 508 pp., 1914. XVII–XVIII: Commentationes analyticæ ad theoriam integralium pertinentes. Volumen 1, edidit A. Gutzmer, 8 + 467 pp., 1915. Volumen 2, ediderunt A. Gutzmer et A. Liapounoff, 12 + 475 pp., 1920.

FUBINI (G.) e VIVANTI (G.). Esercizi di analisi matematica. (Calcolo infinitesimale.) (Grande Biblioteca Tecnica, No. 13.) Torino, Società Tipografico-Editrice Nazionale, 1920. 8vo. 8 + 575 pp. L. 58.00

GATES (S. B.). Pure mathematics for engineers. With an introduction by H. A. Webb. 2 volumes. London, Hodder and Stoughton, 1920. 11 + 191 + 11 + 179 pp. 4s. 6d. + 4s. 6d.

GUTZMER (A.). See EULER (L.).

HAHN (H.). Theorie der reellen Funktionen. 1ter Band. Berlin, Springer, 1921. 8vo. 8 + 600 pp.

HALER (P. J.) and STUART (A. H.). A second course in mathematics for technical students. London, University Tutorial Press, 1920. 8 + 363 pp. 6s.

HEFFTER (L.). Die Grundlagen der Geometrie als Unterbau für die analytische Geometrie. Leipzig, Teubner, 1921. 8vo. 27 pp.

H. C. S. See BIBLIOTHECA.

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